Construction of links from Thompson's group

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Thompson's group F. Definition.

Introduced by Richard Thompson in 1965.

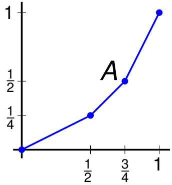
Definition. An element $f \in F$ is a piecewise linear continuous invertible function $f : [0,1] \to [0,1]$ satisfying the conditions

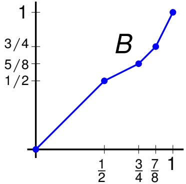
- $f(0) = 0$ and $f(1) = 1$;
- linear except at finitely many dyadic rational numbers;
- on intervals of linearity the derivatives are powers of 2.

Examples.

$$A(x) := \begin{cases} \frac{x}{2}, & 0 \leqslant x \leqslant \frac{1}{2} \\ x - \frac{1}{4}, & \frac{1}{2} \leqslant x \leqslant \frac{3}{4} \\ 2x - 1, & \frac{3}{4} \leqslant x \leqslant 1 \end{cases}$$

$$B(x) := \begin{cases} x, & 0 \leqslant x \leqslant \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4}, & \frac{1}{2} \leqslant x \leqslant \frac{3}{4} \\ x - \frac{1}{8}, & \frac{3}{4} \leqslant x \leqslant \frac{7}{8} \\ 2x - 1, & \frac{7}{8} \leqslant x \leqslant 1 \end{cases}$$

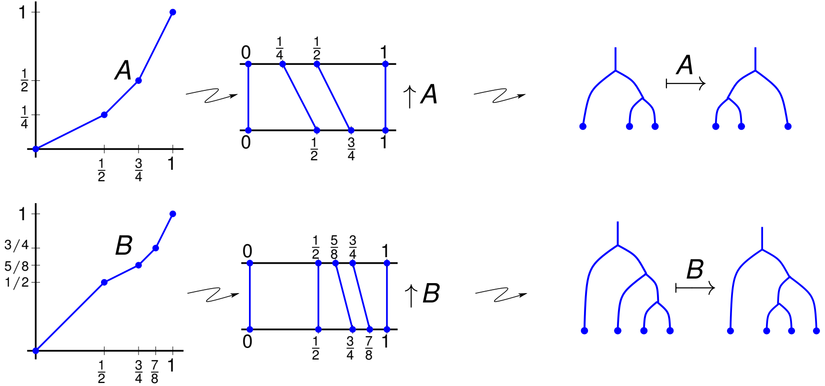




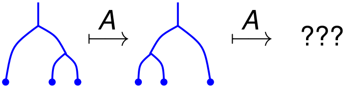
J. W. Cannon, W. J. Floyd, W. R. Parry, *Introductory notes on Richard Thompson's groups*, L'Enseignement Mathématique **42** (1996) 215–256.

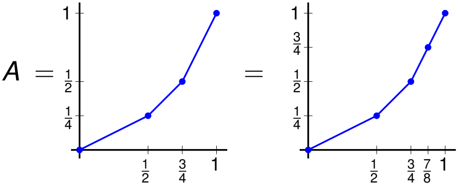
Theorem. *F has the finite presentation*

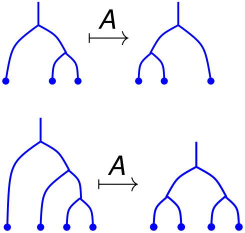
$$\langle A, B \mid [AB^{-1}, A^{-1}BA], [AB^{-1}, A^{-2}BA^{2}] \rangle$$



Thompson's group F. Calculations.

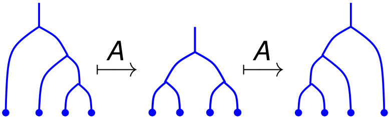






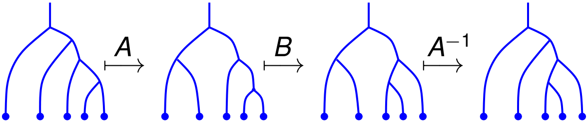
Adding a *caret*, , to the corresponding vertices of both trees does not change the element of the Thompson group.

Thompson's group F. Calculations.

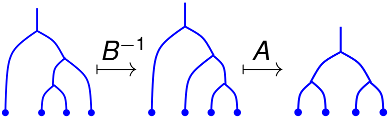


Now we can do calculations.

$A^{-1}BA$:

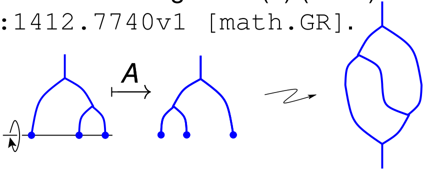


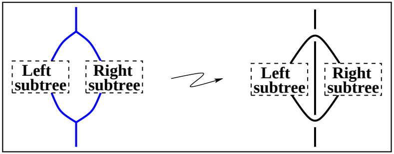
AB^{-1} :

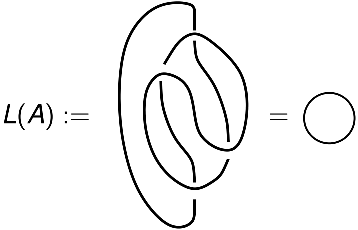


Jones' construction of links from elements of F.

V.Jones, *Some unitary representations of Thompson's groups F and T*, Journal of Combinatorial Algebra **1**(1) (2017) 1–44.
Preprint `arXiv:1412.7740v1 [math.GR]`.



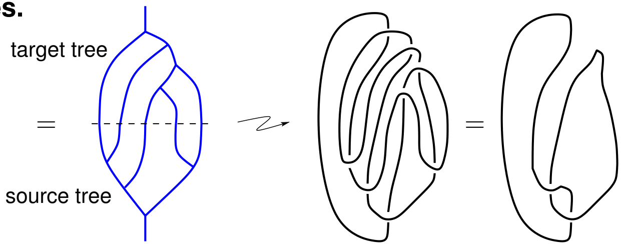




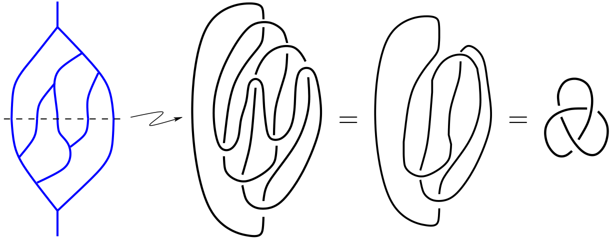
Thompson's group links.

Examples.

$A^{-1}BA^{-1} =$



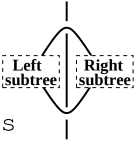
$BA^{-1}B^{-1}A^{-1} =$



Jones' theorem.

V. F. R. Jones, *On the construction of knots and links from Thompson's groups*, in *Knots, low-dimensional topology and applications*, Springer Proc. Math. Stat. **284**, Springer (2019) 43–66. Preprint `arXiv:1810.06034v1 [math.GT]`.

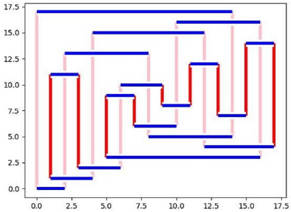
Theorem. *For any link diagram D there is an element $g \in F$, such that the diagram $L(g)$ is isotopic to D.*

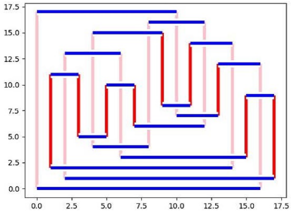


$L(g) = \overline{L(g^{-1})}$

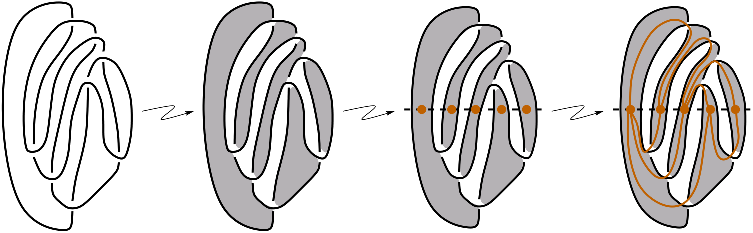
Dennis Sweeney: Borromean rings

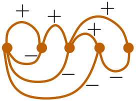
`https://github.com/sweeneyde/thompson_knots`

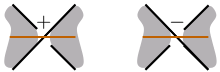




Signed Tait graph.





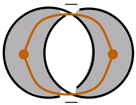


A signed Tait graph is in the *Thompson position* if

- all its vertices are on the x-axis
- all positive edges are in the upper half-plane and all negative edges are in the lower half-plane
- each half-plane's edges form a directed tree rooted at the leftmost vertex, with edges directed left to right.

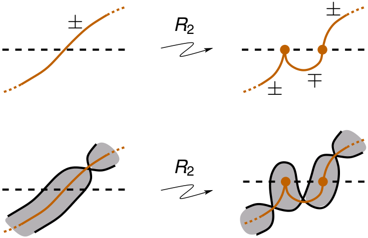
Jones' algorithm.

Idea. Start with arbitrary signed Tait graph and transform it to the Thompson position by Reidemeister moves.



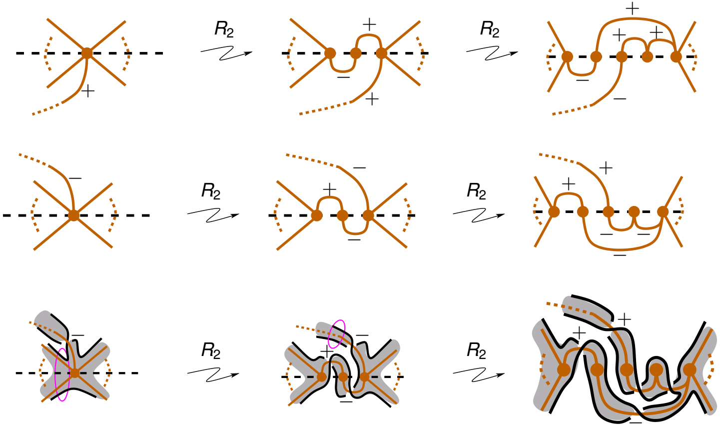
Step 1. Move all the vertices of the Tait graph on the *x*-axis.

Step 2. For any edge that crosses the *x*-axis, replace it with some that don't.



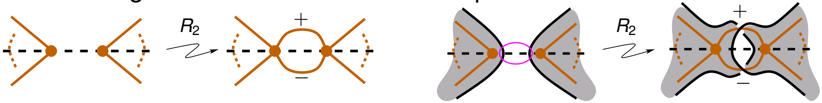
Jones' algorithm.

Step 3. For each positive edge on the bottom or negative edge on the top, make the correct sign.

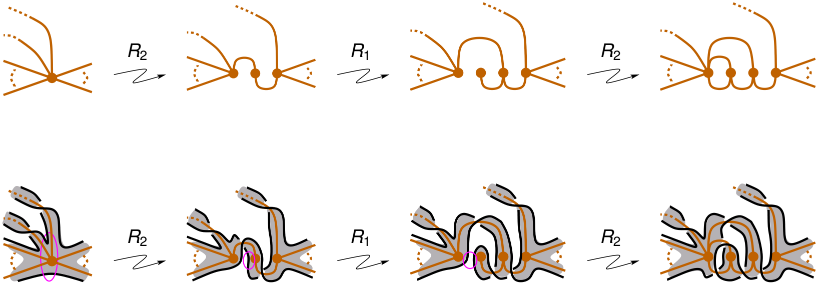


Jones’ algorithm.

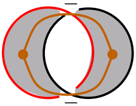
Step 4. Ensure that each vertex (except the leftmost one) has an incident edge from the left on either half plane.

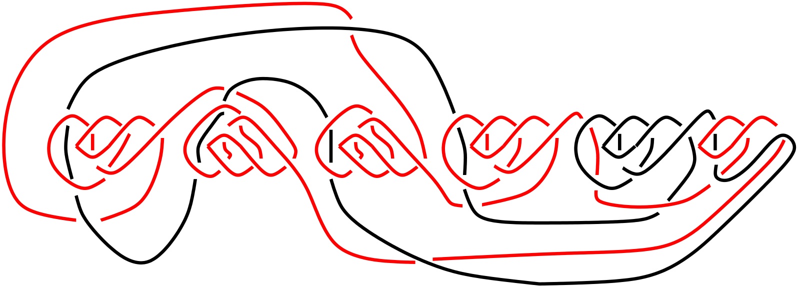


Step 5. Ensure that each vertex has only one incident edge from the left on either half plane.

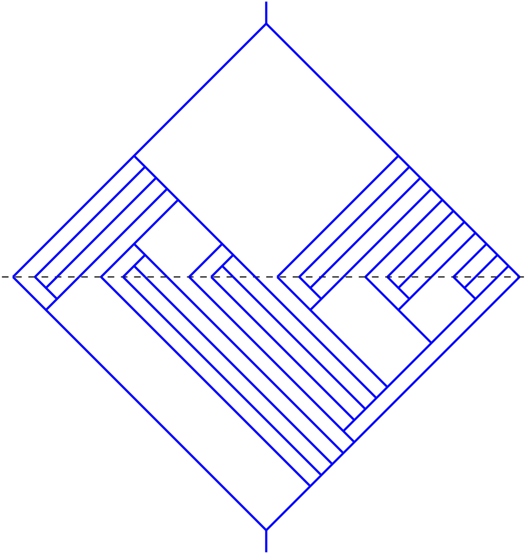


Jones' algorithm. Hopf link.

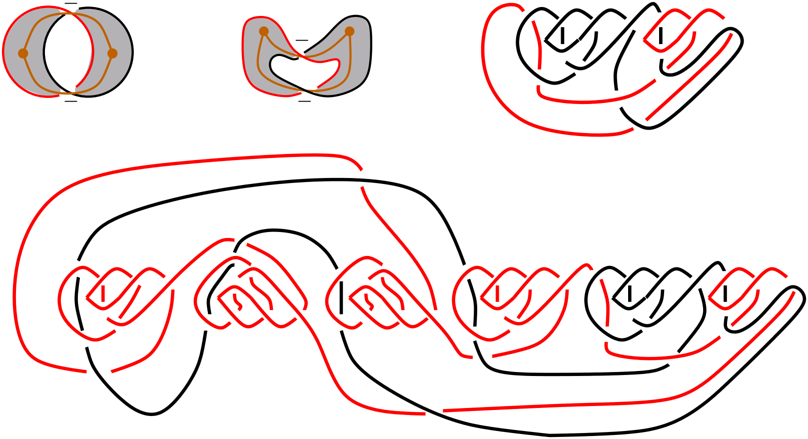




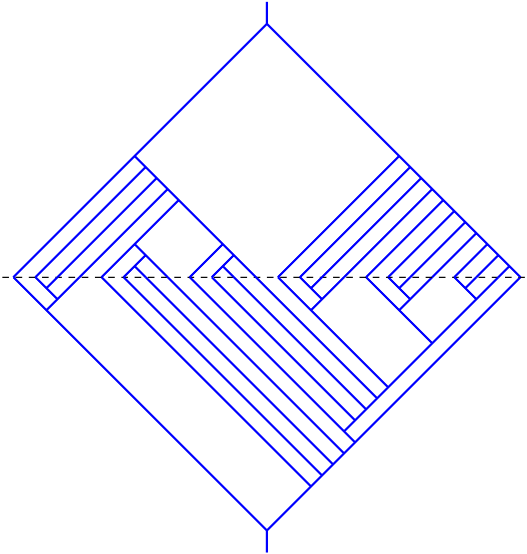
Jones' algorithm. Hopf link. Thompson element.

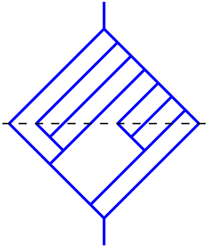


Jones' algorithm. Hopf link, better diagram.



Jones' algorithm. Hopf link, better Thompson element.





Future research.

- Find an efficient algorithm to construct a Thompson element from a link diagram. Improve the Jones algorithm.
- Calculate the Thompson index of first knots from the knot table.
- Find an upper bound for the Thompson index $t(L)$ in terms of the crossing number $c(L)$ similar to the arc index. Is it true that $t(L) \leqslant c(L) + const$?

Knots and Graphs 2021:

`https://people.math.osu.edu/chmutov.1/wor-gr-su21/wor-gr.htm`

THANK YOU!!!