Partial Duality of Hypermaps

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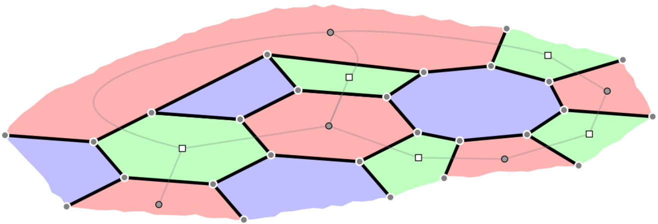
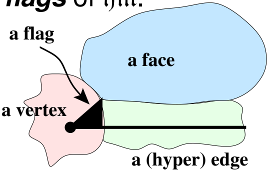
Hypermaps.

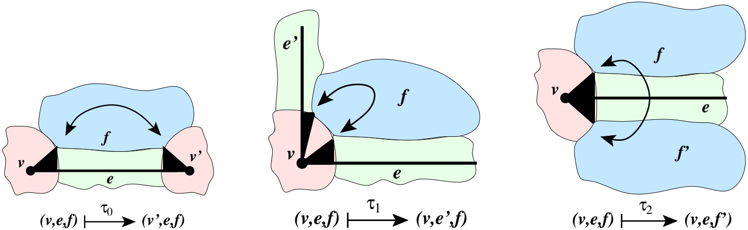


Local view of a hypermap. Vertices are red, hyperedges are green, faces are blue.

τ-model (bi-rotation system).

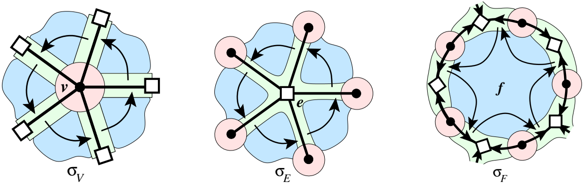
A hypermap $\mathfrak{hm}$ is a triple of fixed point free involutions, (τ_0, τ_1, τ_2), acting on a set of ***local flags*** of $\mathfrak{hm}$.



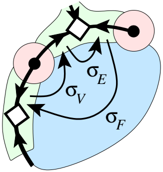


σ-model (rotation system).

A *half-edge* is a small part of a hyperedge near a vertex incident to this hyperedge.
An oriented hypermap is a triple of permutations $(\sigma_V, \sigma_E, \sigma_F)$ of its *half-edges* satisfying the relation $\sigma_F\sigma_E\sigma_V = 1$.

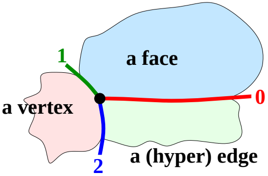


$\sigma_F\sigma_E\sigma_V = 1$:



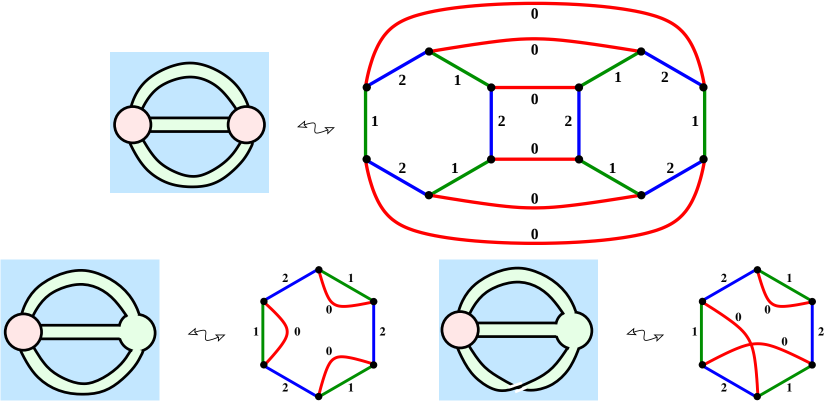
[2]-colored graphs.

The boundaries of cells of a hypermap form a 3-regular graph embedded into the surface of the hypermap.
Edge coloring: the arcs of intersection of hyperedges and faces are colored by 0, the arcs of intersection of vertices and faces are colored by 1, and the arcs of intersection of vertices and hyperedges are colored by 2.



The vertices corresponds to (local) flags of $\mathfrak{hm}$ and its edges of color i correspond to the orbits of the involution τ_i.

[2]-colored graphs. Examples.

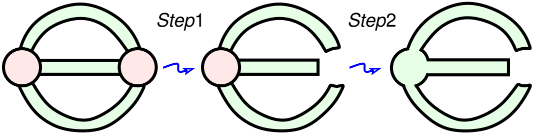


A hypermap corresponding to a [2]-colored graph Γ is orientable if and only if Γ is bipartite.

Partial duality. Construction steps 1 & 2.

Let $\mathfrak{hm}$ be a hypermap and S be a subset of cells of $\mathfrak{hm}$ of the same type, say vertex-cells. We construct the *partial dual hypermap* $\mathfrak{hm}^S$ relative to S. Choose a different type of cells, say hyperedges; the resulting hypermap does not depend on this choice.

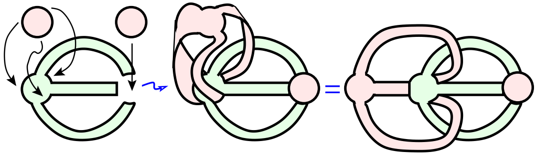
Step 1. Consider the boundary ∂F of the surface F which is the union of the cells from S and all cells of the chosen type, hyperedges in our case.



Step 2. Glue a disk to each connected component of ∂F. These will be the ***hyperedge-cells*** for $\mathfrak{hm}^S$.

Partial duality. Construction step 3.

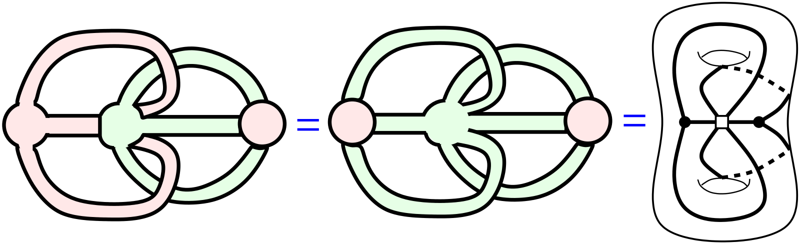
Step 3. Take a copy of every vertex. These disks will be the ***vertex-cells*** for $\mathfrak{hm}^S$.



Every vertex disk of the original hypermap $\mathfrak{hm}$ contributes one or several intervals to ∂F.
If a vertex belong to S, then it contributes to F itself and a part of its boundary contributes to ∂F.
If a vertex is not in S, then it has some hyperededges attached to it. So such a vertex-disk has a common boundary intervals with F and therefore contributes these intervals to ∂F.
The new copies of the vertex-disks, as vertices of $\mathfrak{hm}^S$, are attached to new hyperedges exactly along the same intervals as the old ones.

Partial duality. Construction step 4.

Step 4. At the previous steps we constructed the vertex and hyperedge cells for the partial dual $\mathfrak{hm}^S$.
Their union forms a surface with boundary. Glue a disk to each of its boundary components. These are going to be the ***faces*** of $\mathfrak{hm}^S$.



Partial duality in σ-model.

Theorem.
Let $E' \subseteq E$ be a subset of hyperedges of a hypermap $\mathfrak{hm} = (\sigma_V, \sigma_E, \sigma_F)$ given in σ-model, $\sigma_F\sigma_E\sigma_V = 1$. Then its partial dual is given by the permutations

$$\mathfrak{hm}^{E'} = (\sigma_{E'}\sigma_V,\ \sigma_{\overline{E'}}\sigma_{E'}^{-1},\ \sigma_F\sigma_{E'})\ ,$$

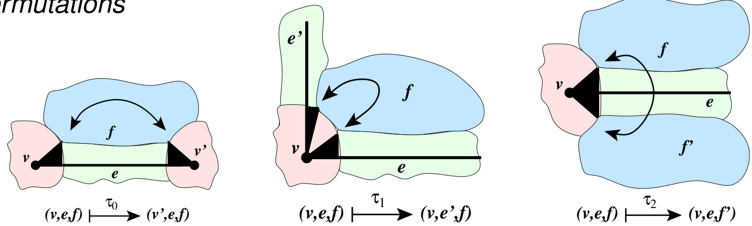
where $\sigma_{E'}$ denotes the permutation consisting of the cycles corresponding to the elements of E' respectively, and overline means the complementary set of cycles.

In particular case of maps (or ribbon graphs) all cycles of the permutation σ_E are transpositions. So $\sigma_{E'}^{-1} = \sigma_{E'}$ and $\sigma_{\overline{E'}}\sigma_{E'}^{-1} = \sigma_E$.

Section 5.2 of [J. L. Gross, T. Mansour, T. W. Tucker], *Partial duality for ribbon graphs, I: Distributions*, European J. Combin. **86** (2020) #103084, 1–20.

Partial duality in τ-model.

Theorem. *Consider the τ-model of a hypermap $\mathfrak{hm}$ given by the permutations*



Let $E' \subseteq E$ be a subset of hyperedges, $\tau_0^{E'}$ be the product of all transpositions in τ_0 for $e \in E'$, and $\tau_2^{E'}$ be the product of all transpositions in τ_2 for $e \in E'$. Then its partial dual $\mathfrak{hm}^{E'}$ is given by the permutations

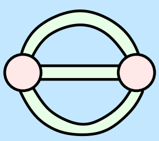
$$\tau_0(\mathfrak{hm}^{E'}) = \tau_0\tau_0^{E'}\tau_2^{E'}, \qquad \tau_1(\mathfrak{hm}^{E'}) = \tau_1, \qquad \tau_2(\mathfrak{hm}^{E'}) = \tau_1\tau_0^{E'}\tau_2^{E'}\ .$$

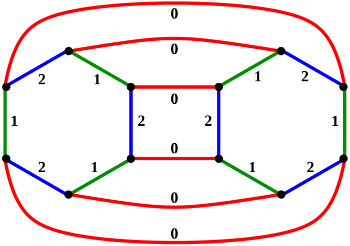
In other words the permutations τ_0 and τ_2 swap their transpositions of local flags around the edges in E'.

Partial duality for [2]-colored graphs.

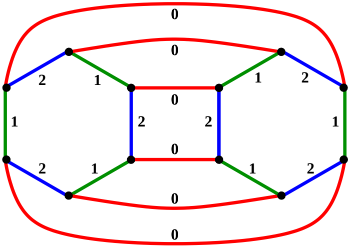
Theorem. *Let* $\Gamma_{\mathfrak{hm}}$ *be a* $[2]$*-colored graph corresponding to a hypermap* $\mathfrak{hm}$*,* $V' \subseteq V$ *be a subset of vertices, and* $\mathfrak{S}$ *be the subset of* 12*-cycles of* $\Gamma_{\mathfrak{hm}}$ *corresponding to the vertices of* V'*.Then the* $[2]$*-coloured graph* $\Gamma_{\mathfrak{hm}^{V'}}$ *of the partial dual hypermap* $\mathfrak{hm}^{V'}$ *is obtained from* $\Gamma_{\mathfrak{hm}}$ *by swapping the colours 1 and 2 for all edges in the* 12*-cycles of* $\mathfrak{S}$*.*

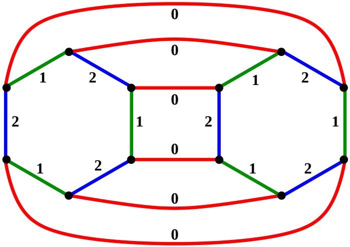
Corollary. *The underlying graphs of* $\Gamma_{\mathfrak{hm}}$ *as an abstract uncolored graph is invariant under the partial duality.*

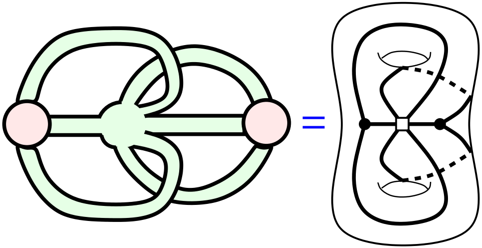
$\mathfrak{hm} =$ 

$\Gamma_{\mathfrak{hm}} =$ 

Partial duality for [2]-colored graphs. Example.

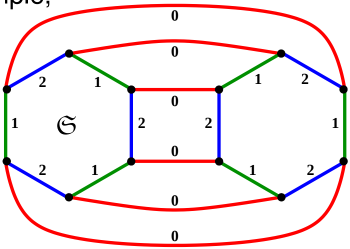
$\Gamma_{\mathfrak{hm}} =$ 

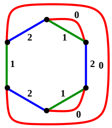
$\Gamma_{\mathfrak{hm}^{V'}} =$ 

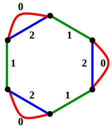
$\Gamma_{\mathfrak{hm}^{V'}} =$ 

Change of genus.

Let $\Gamma_{\mathfrak{hm}}$ be the [2]-colored graph corresponding to a hypermap $\mathfrak{hm}$, $\mathfrak{S}$ be a 12-cycle corresponding to a vertex-cell of $\mathfrak{hm}$ relative to which we are going to do a partial duality.
We define two reduced [2]-colored graphs Γ_{10} and Γ_{20} as follows.
First for Γ_{10}, we consider a subgraph of $\Gamma_{\mathfrak{hm}}$ formed by the cycle $\mathfrak{S}$ and all 10-cycles incident with $\mathfrak{S}$. Then we contract all the 1-edges not in $\mathfrak{S}$, so that every 10-path outside $\mathfrak{S}$ will be replaced by a single edge of color 0.
In our example,

$\Gamma_{\mathfrak{hm}} =$ 

$\Gamma_{10} =$ 

$\Gamma_{20} =$ 

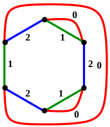
Change of genus.

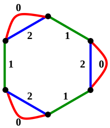
Theorem. *The* ***Euler genus*** $\gamma(\mathfrak{hm})$ *changes according to the formula.*

$$\gamma(\mathfrak{hm}^{\mathfrak{S}}) - \gamma(\mathfrak{hm}) = B^{\{10\}}(\Gamma_{10}) - B^{\{20\}}(\Gamma_{10}) + B^{\{20\}}(\Gamma_{20}) - B^{\{10\}}(\Gamma_{20}),$$

where $B^{\{tk\}}(\Gamma_{ik})$ *is the number of* tk*-cycles of* Γ_{ik}.

In our example,

$\Gamma_{10} =$ 

$\Gamma_{20} =$ 

$$B^{\{10\}}(\Gamma_{10}) = 3, \quad B^{\{20\}}(\Gamma_{10}) = 1, \quad B^{\{20\}}(\Gamma_{20}) = 3, \quad B^{\{10\}}(\Gamma_{20}) = 1.$$

Therefore, $\gamma(\mathfrak{hm}^{\mathfrak{S}}) - \gamma(\mathfrak{hm}) = 3 - 1 + 3 - 1 = 4.$

THANK YOU!!!