# Twist polynomial for delta-matroids

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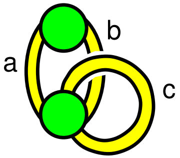
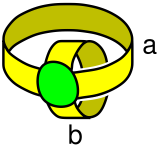
| Matroids | Δ-matroids [A. Bouchet, 1987] |
| --- | --- |
| *A matroid* is a pair $M = (E, \mathcal{B})$ consisting of a finite set $E$ and a nonempty collection $\mathcal{B}$ of its subsets, called *bases*, satisfying the axioms:<br>**(B1)** *No proper subset of a base is a base.*<br>**(B2)** (**Exchange axiom**) *If $B_1$ and $B_2$ are bases and $b_1 \in B_1 - B_2$, then there is an element $b_2 \in B_2 - B_1$ such that $(B_1 - b_1) \cup b_2$ is a base.* | *A Δ-matroid* is a pair $M = (E; \mathcal{F})$ consisting of a finite set $E$ and a nonempty collection $\mathcal{F}$ of its subsets, called *feasible sets*, satisfying the<br>**Symmetric Exchange axiom**<br>*If $F_1$ and $F_2$ are two feasible sets and $f_1 \in F_1 \Delta F_2$, then there is an element $f_2 \in F_1 \Delta F_2$ such that $F_1 \Delta \{f_1, f_2\}$ is a feasible set.* |

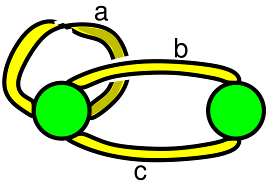
A *quasi-tree* is ribbon graph $\mathbb{G}$ with a single boundary component.




Spanning quasi-trees:
$\{a\}, \{b\}, \{a, b, c\}$




Spanning quasi-trees:
$\emptyset, \{a, b\}$




Spanning quasi-trees:
$\{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}$

**Theorem.** *Let* $\mathbb{G} = (V, E)$ *be a ribbon graph. Then*

$$D(\mathbb{G}) := (E; \{\textit{spanning quasi-trees}\})$$

*is a* Δ*-matroid.*

Let $C$ be a symmetric $|E| \times |E|$ matrix over $\mathbb{F}_2$, with rows and columns indexed by the elements of $E$.

**Theorem.**
$$D(C) := (E; \{F \subseteq E | C[F] \text{ is non-singular}\})$$
*is a $\Delta$-matroid.*

**Example.** Let $C := A_G$ be the adjacency matrix of an abstract graph $G$ and $E$ is the set of its vertices.

If $G = K_n$ is the complete graph with $n$ vertices, then the feasible sets of the corresponding $\Delta$-matroid $D_n := D(A_{K_n})$ and all subsets of $E$ of even cardinality.
$D_3 = (\{1, 2, 3\}; \{\emptyset, \{1, 2\}, \{1, 3\}, \{2, 3\}\})$

If $G$ is a graph od a single vertex and a single loop attached to it, then its matroid is $N_1 := D(A_G) = (\{1\}; \{\emptyset, \{1\}\})$. The corresponding ribbon graph $\mathbb{G}$ consist of a sinle vertex and a single loop-ribbon half-twisted.

Let $D = (E; \mathcal{F})$ be a Δ-matroid and $e \in E$.
$e$ is a *loop* iff $\forall F \in \mathcal{F}, e \notin F$.  $e$ is a *coloop* iff $\forall F \in \mathcal{F}, e \in F$.

If $e$ is not a loop, $D/e := (E \setminus \{e\}; \{F \setminus \{e\} | F \in \mathcal{F}, e \in F\})$.

If $e$ is not a coloop, $D \setminus e := (E \setminus \{e\}; \{F | F \in \mathcal{F}, F \subset E \setminus \{e\}\})$.

**Twists of Δ-matroids.** Let $D = (E; \mathcal{F})$ be a Δ-matroids and $A \subseteq E$.

$$D * A := (E; \{F \Delta A | F \in \mathcal{F}\}).$$

**Dual Δ-matroid:** $D^* := D * E$.

**Theorem.** $D(\mathbb{G}) * A = D(\mathbb{G}^A)$.

A Δ-matroid is binary if it is a twist of a representative (over $\mathbb{F}_2$) Δ-matroid.

Let $D = (E, \mathcal{F})$ be a Δ-matroid.
$D_{min} := (E, \mathcal{F}_{min})$, where $\mathcal{F}_{min} := \{F \in \mathcal{F} | F \text{ is of minimal possible cardinality}\}$.

$D_{max} := (E, \mathcal{F}_{max})$, where $\mathcal{F}_{max} := \{F \in \mathcal{F} | F \text{ is of maximal possible cardinality}\}$.

**Properties.**

- $D_{min}$ and $D_{max}$ are matroids. *Width* $w(D) := r(D_{max}) - r(D_{min})$
- $w(D(\mathbb{G})) = 2g(\mathbb{G})$, the genus of $\mathbb{G}$.
- $(D(\mathbb{G}))_{min} = \mathcal{C}(\mathbb{G})$.  $(D(\mathbb{G}))_{max} = (\mathcal{C}(\mathbb{G}^*))^*$.
- $D(\mathbb{G}) = \mathcal{C}(\mathbb{G})$ iff $\mathbb{G}$ is a planar ribbon graph.

**The twist polynomial of a delta-matroid** $D = (E, \mathcal{F})$ is the generating function for the width of all twists of $D$,

$$^{\partial}w_D(z) := \sum_{A \subseteq E} z^{w(D*A)}$$

J. L. Gross, T. Mansour, T. W. Tucker, *Partial duality for ribbon graphs, I: Distributions*, European Journal of Combinatorics **86** (2020) 103084, 1–20:
**GMT-conjecture.** *For any ribbon graph there is a subset of edges partial duality relative to which changes the genus.*

Q. Yan, X. Jin, *Counterexamples to a conjecture by Gross, Mansour, and Tucker on partial-dual genus polynomials of ribbon graphs*, European Journal of Combinatorics **93** (2021) 103285:
**Theorem.** *The genus of any partial dual to $B_{2n+1}$ is equal to $n$.*
$D(B_{2n+1}) = D_{2n+1}$. In particular, ${}^{\partial}w_{D_{2n+1}}(z) = 2^{2n+1}z^{2n}$.

—, F. Vignes-Tourneret, *On a conjecture of Gross, Mansour and Tucker*, European Journal of Combinatorics **97**(3) (2021) 103368:
**Theorem.** *For any join-prime ribbon graph different from partial duals of $B_{2n+1}$, there are partial duals of different genus.*

Independent proofs and non-orientable case:

- Maya Thompson (Royal Holloway University of London).
- Q. Yan, X. Jin, *Partial-dual genus polynomials and signed intersection graphs*, Forum of Mathematics, Sigma **10** (2022) 1–16.

Q. Yan, X. Jin, *Twist monomials of binary delta-matroids*. Preprint arXiv:2205.03487v1 [math.CO]:
**Theorem.** *A normal binary delta-matroid has a twist monomial iff each connected component of its corresponding looped simple graph is either a complete graph of odd order or a single vertex with a loop.*

| $\lvert E\rvert$ | Number of twist nonequivalent binary $\Delta$-matroids on $E$ | Number of twist nonequivalent $\Delta$-matroids on $E$ |
|---|---|---|
| 2 | 5 | 5 |
| 3 | 13 | 16 |
| 4 | 40 | 90 |
| 5 | 141 | 2902 |

D. Yuschak, *Delta-matroids with twist monomials*. Preprint arXiv:2208.13258v1 [math.CO] 28 Aug 2022:
**Theorem.** *If a $\Delta$-matroid has a twist monomial, then it is binary.*

Thus the only $\Delta$-matroids with twist monomials are

$$\boxed{D_{2n_1+1} \oplus \cdots \oplus D_{2n_k+1} \oplus N_1 \oplus \cdots \oplus N_1}$$

A. Bouchet, A. Duchamp, *Representability of $\Delta$-matroids over $GF(2)$*, Linear Algebra Appl. **146** (1991) 67–78:

**Theorem.** *A $\Delta$-matroid is binary iff it has no minor isomorphic to one of the following delta-matroids:*

*1.* $(\{1,2,3\},\{\emptyset,\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\})$

*2.* $(\{1,2,3\},\{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\})$

*3.* $(\{1,2,3\},\{\emptyset,\{2\},\{3\},\{1,2\},\{1,3\},\{1,2,3\}\})$

*4.* $(\{1,2,3,4\},\{\emptyset,\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\})$

*5.* $(\{1,2,3,4\},\{\emptyset,\{1,2\},\{1,4\},\{2,3\},\{3,4\},\{1,2,3,4\}\})$

**THANK YOU!!!**