Partial duality for ribbon graphs

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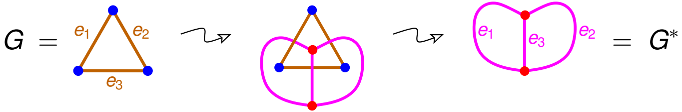
MATRIX Workshop on Uniqueness and Discernment in Graph Polynomials

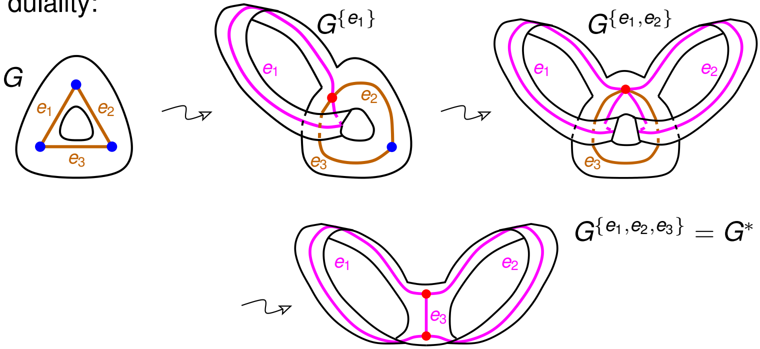
October 16—27, 2023

Partial duality. Example.

Classical *Euler-Poincaré* duality for graphs on surfaces:



Partial dulality:

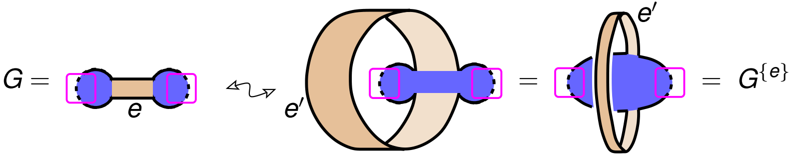


For a ribbon graph G and a subset of the edge-ribbons $A \subseteq E(G)$, the ***partial dual***, G^A of G relative to A is a ribbon graph constructed as follows.

- The vertex-discs of G^A are bounded by connected components of the boundary of the spanning subgraph of G containing all the vertices of G and only the edges from A.
- The edge-ribbons of $E(G) \setminus A$ are attached to these new vertices exactly at the same places as in G. The edge-ribbons from A become parts of the new vertex-discs now. For $e \in A$ we take a copy of e, e', and attach it to the new vertex-discs in the following way. The rectangle representing e intersects with vertex-discs of G by a pair of opposite sides. But it intersects the boundary of the spanning subgraph, that is the new vertex-discs, along the arcs of the other pair of its opposite sides. We attach e' to these arcs by this second pair. The copies of the first pair of sides in e' become the arcs of the boundary of G^A.

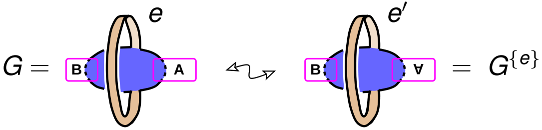
Partial duality relative to a set of edges can be done step by step one edge at a time.

The partial duality relative to one edge:



Here the boxes with dashed arcs mean that there might be other edges attached to these vertices.

The partial duality relative to a non-orientable loop:



Lemma. *Let G be a ribbon graph.*

(a) *Suppose* $E(G) \ni e \notin A \subseteq E(G)$*. Then* $G^{A\cup\{e\}} = \left(G^A\right)^{\{e\}}$*.*

(b) $\left(G^A\right)^A = G$*.*

(c) $\left(G^A\right)^B = G^{\Delta(A,B)}$*, where* $\Delta(A,B) := (A\cup B)\setminus(A\cap B)$ *is the symmetric difference of sets.*

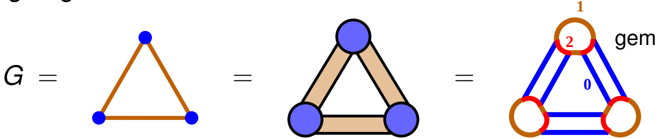
(d) *The partial preserves orientability of ribbon graphs.*

(e) *Let* $\widetilde{G}$ *be a surface without boundary obtained from G by gluing discs to all boundary component of G. Then* $\widetilde{G^A} = \widetilde{G^{E(G)\setminus A}}$*.*

(f) *The generalized duality preserves the number of connected components of ribbon graphs.*

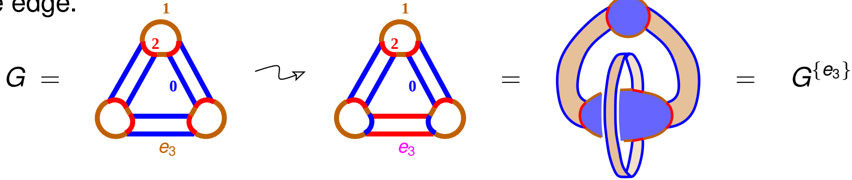
Partial duality for gems.

Definition. A *gem* (graph-encoded map) is a trivalent graph whose edges colored into three colors 0, 1, and 2 in such a way that that at every vertex the three meeting edges have all different colors.



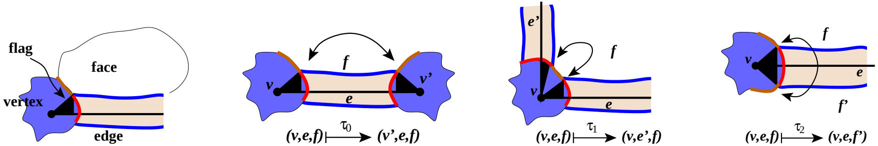
The vertices of a ribbon graph are the 12-cycles of the corresponding gem; the edges are 02-cycles (of length 4); and the faces are 01-cycles.

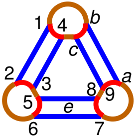
Partial duality relative to an edge is the the swapping colors 1 and 2 along the edge.



Partial duality for bi-rotation systems.

Definition. A *bi-rotation system* is a set of three fixed point free involutions, τ_0, τ_1, and τ_2, acting on a set of vertices of the corresponding gem.



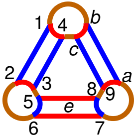
$G =$ 

$$\tau_0(G) = (12)(34)(58)(67)(9c)(ab)$$
$$\tau_1(G) = (1b)(4c)(26)(35)(7a)(89)$$
$$\tau_2(G) = (1c)(23)(56)(78)(9a)(bc)$$

$a{=}10$, $b{=}11$, $c{=}12$
$\tau_0^e{=}(58)(67)$
$\tau_2^e{=}(56)(78)$

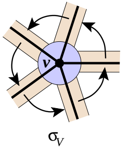
Partial duality relative to an edge e. Let τ_0^e be the product of two transpositions in τ_1 for e, and τ_2^e be the product of two transpositions in τ_2 for e.

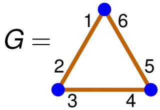
Then $\quad \tau_0(G^{\{e\}}) = \tau_0(G)\tau_0^e\tau_2^e, \quad \tau_1(G^{\{e\}}) = \tau_1(G), \quad \tau_2(G^{\{e\}}) = \tau_2(G)\tau_0^e\tau_2^e.$

$G^{\{e\}} =$ 

$$\tau_0(G^{\{e\}}) = \tau_0(G)\tau_0^e\tau_2^e = (12)(34)(56)(78)(9c)(ab)$$
$$\tau_1(G^{\{e\}}) = (1b)(4c)(26)(35)(7a)(89)$$
$$\tau_2(G^{\{e\}}) = \tau_2(G)\tau_0^e\tau_2^e = (1c)(23)(58)(67)(9a)(bc)$$

Definition. A *rotation system* is cyclic order of half-edges around every vertex and an involution of two half-edges forming a single edge.



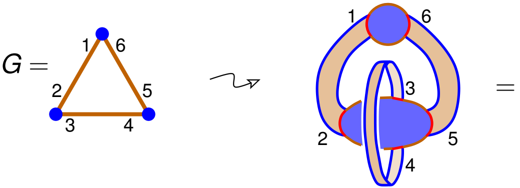


$\sigma_V(G) = (16)(23)(45)$
$\sigma_E(G) = (12)(34)(56)$

Partial duality relative to an edge $e = (ab)$.

$$\sigma_V(G^{\{e\}}) = (ab) \cdot \sigma_V(G), \quad \sigma_E(G^{\{e\}}) = \sigma_E(G)\ .$$

For $e_3 = (34)$, $\sigma_V(G^{\{e_3\}}) = (34) \cdot (16)(23)(45) = (16)(2453)$.



$G^{\{e_3\}}$

The *partial-dual genus polynomial*: J. L. Gross, T. Mansour, T. W. Tucker, *Partial duality for ribbon graphs, I: Distributions*, European Journal of Combinatorics **86** (2020) 103084, 1–20.

$$^{\partial}\Gamma_G(z) := \sum_{A \subseteq E(G)} z^{g(G^A)}$$

Open problem. Are there any relations of the partial-dual genus polynomial with other ribbon graph polynomials?

THANK YOU!!!