Ribbon graph polynomials and weight systems

Sergei Chmutov

Ohio State University, Mansfield

Geometry and Combinatorics of Integrable Systems, Yerevan (Armenia)

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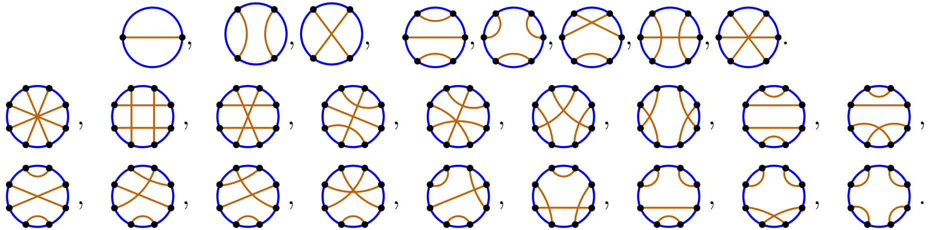
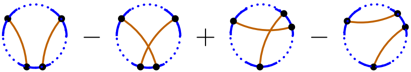
Overview

Chord diagrams

A ***chord diagram*** of order n (or degree n) is an oriented circle with a distinguished set of n disjoint pairs of distinct points, considered up to orientation preserving homeomorphisms of the circle.



Four-term relation for chord diagrams:



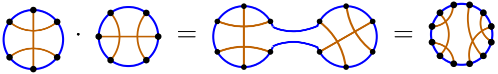
$$= 0. \qquad (4T)$$

$\mathcal{A}_n := Sp_{\mathbb{C}}\langle$chord diagrams with n chords$\rangle/(4T)$

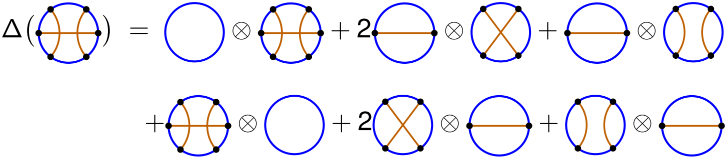
Hopf algebra of chord diagrams and weight systems

$\boxed{\mathcal{A} := \bigoplus_{n=0}^{\infty} \mathcal{A}_n}$ is a (graded, commutative, and cocommutative) Hopf algebra

with ***multiplication:***



and ***comultiplication:***



A ***weight system*** of order n is a function on $\mathcal{A}_n$ satisfying (4T): $\mathcal{W}_n := \mathcal{A}_n^*$.

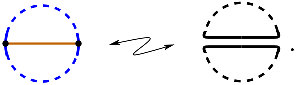
$\boxed{\mathcal{W} := \bigoplus_{n=0}^{\infty} \mathcal{W}_n}$ with ***multiplication:*** $(w_1 \cdot w_2)(D) := (w_1 \otimes w_2)(\Delta(D))$.

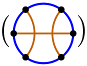
Examples: Unit weight system and $\mathfrak{gl}_N$

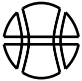
The ***unit weight system*** $\mathbf{I}$ is the weight system that is equal to 1 on every chord diagram. $\mathbf{I}_n$ is the function on chord diagrams which is equal to 1 on any diagram of degree n and 0 on chord diagrams of all other degrees.

$$\mathbf{I}_n \cdot \mathbf{I}_m = \binom{m+n}{n} \mathbf{I}_{n+m}\,, \qquad \mathbf{I}_n = \frac{\mathbf{I}_1^n}{n!}\,, \qquad \mathbf{I} = \sum_{n=0}^{\infty} \mathbf{I}_n = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{I}_1^n = e^{\mathbf{I}_1}$$

M. Kontsevich, D. Bar-Natan: $\mathfrak{gl}_N$ with standard representation.
$\varphi_{\mathfrak{gl}_N}(D) = N^{f(D)}$, where $f(D)$ is the number of connected components of the curve obtained by doubling all chords of a chord diagram D.



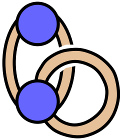
$\varphi_{\mathfrak{gl}_N}($  $) = N^2.$



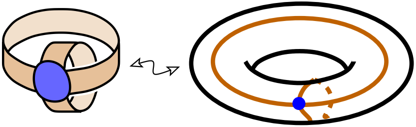
Ribbon graphs

A ***ribbon graph*** is a surface with boundary decomposed into a number of closed topological discs of two types: ***vertex-discs*** and ***edge-ribbons***, satisfying the conditions: the discs of the same type are pairwise disjoint; the vertex-discs and the edge-ribbons intersect by disjoint line segments, each such line segment lies on the boundary of precisely one vertex and precisely one edge, and every edge contains exactly two such line segments, which are not adjacent.

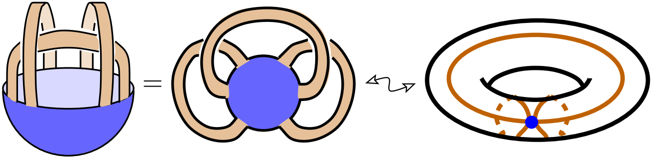
(a)



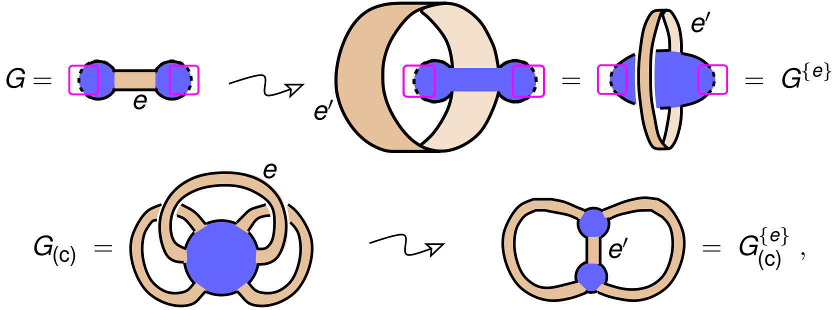
(b)



(c)



Partial duality



J. L. Gross, T. Mansour, and T. W. Tucker: ***Partial-dual genus polynomial.***

$$\boxed{{}^{\partial}\Gamma_G(z) := \sum_{A\subseteq E(G)} z^{g(G^A)}}$$

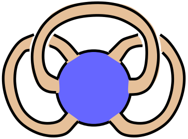
${}^{\partial}\Gamma_{G_{(c)}}(z) = 2 + 6z.$

The Bollobás–Riordan polynomial

$$B_G(X,Y,Z)=\sum_{A\subseteq E(G)} X^{k(A)-k(G)}\,Y^{n(A)}\,Z^{k(A)-f(A)+n(A)},$$

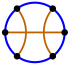
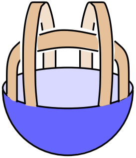
where

- $k(A)$ be the number oA components of A;
- $n(A):=e(A)-v(A)+k(A)$ be the *nullity* of A;
- $f(A)$ be the number of boundary components of A.

$G_{(c)}\ =$ 

$$B_{G_{(c)}}(X,Y,Z)=1+3Y+2Y^2Z^2+Y^2+Y^3Z^2$$

Chord diagrams as ribbon graphs

$D =$  $\rightsquigarrow$ $G_D =$ 

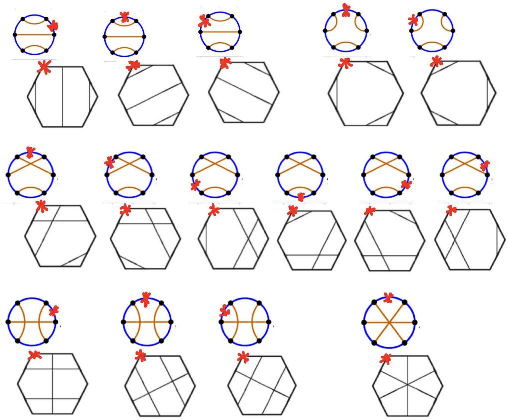
Any function (polynomial valued) on ribbon graphs becomes a function on chord diagrams. Which of them are weight systems?

- The partial-dual genus polynomial, ${}^{\partial}\Gamma_{G_D}(z)$, is a weight system (S.Ch. *Partial-dual genus polynomial as a weight system*, *Communications in Mathematics*, **31**(3) (2023) 113–124).

- The Bollobás–Riordan polynomial, B_{G_D}, is a weight system.

$$B_{G_D}(X, Y, Z) = Z\Big(\mathbf{I} \cdot (YZ)^{\deg D} \varphi_{\mathfrak{gl}_N}\Big)(D)\Big|_{N=Z^{-1}}$$

Harer-Zagier formula



$$T_n(N) \quad := \sum_{\substack{D \\ \text{based chord} \\ \text{diagram with} \\ n \text{ chords}}} N^{f(D)}$$

$$= \sum \varphi_{\mathfrak{gl}_N}(D)$$

$$T(N,s) \quad := 1 + 2Ns + 2s \sum_{n=1}^{\infty} \frac{T_n(N)}{(2n-1)!!} s^n$$

$$= \left(\frac{1+s}{1-s} \right)^N$$

Are there any nice formulas for the generating functions of sum of values of other weight systems?

Iain Moffatt's theorem

British Combinatorial Conference, 2024.

Theorem. *Consider a weight system*

$$\psi(D) := z^{1+\deg D}\varphi_{\mathfrak{gl}_N}(D)\Big|_{N=z^{-1}}$$

Then ${}^{\partial}\Gamma_{G_D}(z^2) = \psi^2(D)$.

Proof.

$$\begin{aligned} 2g(G^A) &= 2 - \big(v(G^A) - e(G) + f(G^A)\big) \\ &= 2 + e(G) - \big(v(G^A) + f(G^A)\big) \end{aligned}$$

But $f(G^A) = f(A^c)$ and $v(G^A) = f(G^{A^c}) = f(A)$
So $2g(G^A) = 2 + e(G) - (f(A) + f(A^c))$
And

$${}^{\partial}\Gamma_{G_D}(z^2) = \sum_{A\subseteq E(G)} z^{2g(G_D^A)} = z^{2+\deg D}\cdot \sum_{A\subseteq E(G)} z^{-f(A)-f(A^c)} = \psi^2(D)$$