Moffatt's theorem for delta-matroids

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Iain Moffatt's theorem: British Combinatorial Conference, 2024.

Delta-matroids:	Bollobás–Riordan polynomial $\xrightarrow{???}$ Twist polynomial
Ribbon graphs:	Bollobás–Riordan polynomial $\xrightarrow{???}$ Partial dual genus polynomial
Chord diagrams:	$\mathfrak{gl}_N$ weight system $\xrightarrow{\text{I.Moffatt}}$ Partial dual genus polynomial

Matroids	Δ-matroids [A. Bouchet, 1987]
A matroid is a pair $M = (E, \mathcal{B})$ consisting of a finite set E and a nonempty collection $\mathcal{B}$ of its subsets, called *bases*, satisfying the axioms: **(B1)** *No proper subset of a base is a base.* **(B2)** (**Exchange axiom**) *If B_1 and B_2 are bases and $b_1 \in B_1 - B_2$, then there is an element $b_2 \in B_2 - B_1$ such that $(B_1 - b_1) \cup b_2$ is a base.*	*A Δ-matroid* is a pair $D = (E; \mathcal{F})$ consisting of a finite set E and a nonempty collection $\mathcal{F}$ of its subsets, called *feasible sets*, satisfying the **Symmetric Exchange axiom** *If F_1 and F_2 are two feasible sets and $f_1 \in F_1 \Delta F_2$, then there is an element $f_2 \in F_1 \Delta F_2$ such that $F_1 \Delta \{f_1, f_2\}$ is a feasible set.*

Let $D = (E; \mathcal{F})$ be a Δ-matroid and $e \in E$.
e is a *loop* iff $\forall F \in \mathcal{F}, e \notin F$. e is a *coloop* iff $\forall F \in \mathcal{F}, e \in F$.

If e is not a loop, $D/e := (E \setminus \{e\}; \{F \setminus \{e\} | F \in \mathcal{F}, e \in F\})$.

If e is not a coloop, $D \setminus e := (E \setminus \{e\}; \{F | F \in \mathcal{F}, F \subset E \setminus \{e\}\})$.

Twists of Δ-matroids. Let $D = (E; \mathcal{F})$ be a Δ-matroids and $A \subseteq E$.

$$D * A := (E; \{F \Delta A | F \in \mathcal{F}\}).$$

$D_{min} := (E; \mathcal{F}_{min})$, where $\mathcal{F}_{min} := \{F \in \mathcal{F} | F$ is of minimal possible cardinality$\}$.

$D_{max} := (E; \mathcal{F}_{max})$, where $\mathcal{F}_{max} := \{F \in \mathcal{F} | F$ is of maximal possible cardinality$\}$.

$w(D) := r_{D_{\max}}(E) - r_{D_{\min}}(E)$.

The twist polynomial of a Δ-matroid $D = (E, \mathcal{F})$ is the generating function for the width of all twists of D,

$$^{\partial}w_D(z) := \sum_{A \subseteq E} z^{w(D*A)}$$

Example. Let $D = (E; \mathcal{F})$ where $E = \{a, b\}$, and $\mathcal{F} = \{\emptyset, \{b\}, \{a, b\}\}$.Then
$D * \{a\} = (E; \{\{a\}, \{a, b\}, \{b\}\})$,
$D * \{b\} = (E; \{\{b\}, \emptyset, \{a\}\})$, and
$D * \{a, b\} = (E; \{\{a, b\}, \{a\}, \emptyset\})$.

Therefore

$$^{\partial}w_D(z) = z^2 + z + z + z^2 = 2z^2 + 2z \; .$$

C.Chun, I.Moffatt, S.Noble, R.Rueckriemen, *Matroids, delta-matroids and embedded graphs*, JCTA 2019.

$$BR_D(\{x_e\},\{y_e\},X,Y,Z) := \sum_{A\subseteq E}(\prod_{e\in A} x_e)(\prod_{e\notin A} y_e)X^{r_{D_{\min}}(E)-r_{D_{\min}}(A)}Y^{n_{D_{\min}}(A)}Z^{w(D|_A)},$$

where the width of $D|_A$ on the ground set A is equal to
$w(D|_A) := r_{(D|_A)_{\max}}(A) - r_{(D|_A)_{\min}}(A)$.

Example. $D = (E = \{a,b\}; \mathcal{F} = \{\emptyset, \{b\}, \{a,b\}\})$.
Then

$$BR_D = y_1y_2 + x_1y_2Y + y_1x_2YZ + x_1x_2Y^2Z^2\ .$$

Theorem.

$$[x_1 \dots x_n y_1 \dots y_n] BR^2_D \Big|_{\substack{X=Y=z^{-1} \\ Z=z}} = z^{-|E|} \cdot {}^{\partial}w_D(z)$$

Here the left hand side means the coefficient of BR^2_D at the monimial $x_1 \dots x_n y_1 \dots y_n$. It has to be evaluated at $X = Y = z^{-1}, Z = z$.

Example. $D = (E = \{a, b\}; \mathcal{F} = \{\emptyset, \{b\}, \{a, b\}\})$.

$$[x_1 \dots x_n y_1 \dots y_n] BR^2_D = 2Y^2Z^2 + 2Y^2Z \; .$$

So,

$$[x_1 \dots x_n y_1 \dots y_n] BR^2_D \Big|_{\substack{X=Y=z^{-1} \\ Z=z}} = 2 + 2z^{-1} = z^{-2}(2z^2 + 2z) = z^{-2} \cdot {}^{\partial}w_D(z) \; .$$

- For $\forall A \subseteq E$, $\max\limits_{F\in\mathcal{F}} |A\Delta F| + \min\limits_{F\in\mathcal{F}} |A^c\Delta F| = |E|$.
- For a Δ-matroid $D = \{E;\mathcal{F}\}$,
$$w(D*A) = |E| - \min_{F\in\mathcal{F}} |A\Delta F| - \min_{F\in\mathcal{F}} |A^c\Delta F| .$$
- $w(D*A) = \rho_D(A) + \rho_D(A^c) - |E|$, where $\rho_D(A) := |E| - \min\limits_{F\in\mathcal{F}} |A\Delta F|$ is the Δ-matroid rank function.
- $w(D*A) = w(D|_A) + w(D|_{A^c}) + 2\Big(r_{D_{\min}}(A) + r_{D_{\min}}(A^c) - r_{D_{\min}}(E)\Big)$.
  In particular, if D is a normal Δ-matroids, that is $\emptyset \in \mathcal{F}$, then
  $w(D*A) = w(D|_A) + w(D|_{A^c})$.

Theorem.

$$[x_1 \dots x_n y_1 \dots y_n] BR_D^2 \Big|_{XYZ^2=1} = (XZ)^{r_{D_{\min}}(E) - n_{D_{\min}}(E)} Z^{-|E|} \cdot {}^{\partial}w_D(Z)$$

Pproblems.

- What about the other polynomials except BR_D?
- Are there any substitution into a polynomial which make it invariant under the twists? In particular, the Las Vergnas polynomial.
- The weight systems, i.e. the functions on chord diagrams considered as one-vertex ribbon graphs, are constructed from a Lie algebras and their representations. Is the any way to construct a polynomial for arbitrary ribbon graphs starting from a Lie algebra (+ its representation)? An example is the Bollobás–Riordan polynomial and the $\mathfrak{gl}_N$ weight system.

THANK YOU!!!