Knot theory and Combinatorics

VIGRE Working Group [Summer 2005]

MATH 693, call number 11017-3, 3 credits

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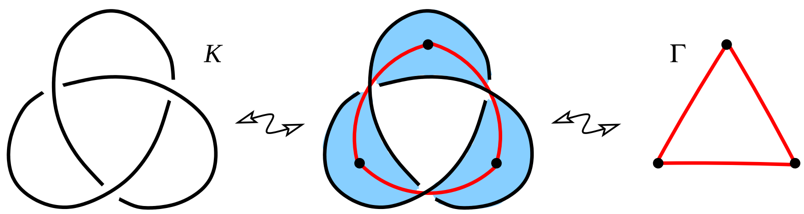
RESEARCH PROJECTS

Virtual knots and ribbon graphs (should be accessible to undergraduates)

There is a remarkable connection between the Kauffman bracket (Jones polynomial) of a link diagram and the Tutte polynomial of graphs that was discovered in [Th, Ka1, Ja]. It works for alternating link diagrams and planar graphs as follows:



Up to the sign and multiplication by a power of t the Jones polynomial $J_K(t)$ equals to the Tutte polynomial $T_\Gamma(-t,-t^{-1})$

It was generalized for non alternating links (and signed graphs) in [Ka2]; and for non-planar graphs (and link diagram on a surface of higher genus) in [CP]. The later case deals with ribbon graphs and the corresponding generalization of the Tutte polynomial discovered by B. Bollobás and O. Riordan in [BR2, BR3]. The link diagrams on a surface are tightly related to virtual knots. About virtual knots and their Kauffman brackets see [Ka3, Kam]. Other correspondences between links and graphs are considered in [Ja, Tr].

Problems

1. Generalize the results of [Ja, Tr] about relations between link invariants and the Tutte polynomial to ribbon graphs and the Bollobás-Riordan polynomial in the spirit of [CP]. Formulate the results in terms of virtual links.

2. Find the relation between Kauffman bracket of a link diagram on a non-orientable surface with the Bollobás-Riordan polynomial of the corresponding non-orientable ribbon graph. This is supposed to be a non-orientable version of [CP].

3. Define a *colored* version of the Bollobás-Riordan polynomial of ribbon graphs like it was done in [BR1, Tr] for the ordinary Tutte polynomial.

4. Define a *ribbon* matroid and all the polynomials for it.

5. The results of [CP] reformulated for virtual links work only for so called *checkerboard colorable* virtual links (see [Kam]). For other links there are no corresponding graphs. What is the right combinatorial object for which we can define Tutte-Bollobás-Riordan like polynomial? Is it a matroid? Ribbon matroid?

6. Consider two different plane embeddings of the same planar graph Γ. Let L_1 and L_2 be the corresponding links. Are they equivalent? This gives a way to construct links with the same Jones polynomial.

7. A *Whitney twist* performed on a graph Γ can be defined as follows [Wh, Hug]. Let Γ_1 and Γ_2 be two graphs. Pick edges $e_1 \in \Gamma_1$ and $e_2 \in \Gamma_2$. Construct a new graph by gluing Γ_1 and Γ_2 along the edges e_1 with e_2 (together with their endpoints) and then removing this edge from the result. In general this can be done in two ways depending how we glue e_1 with e_2. If one of them is Γ then another one is a Whitney twist of Γ.
Whitney proved that any two 2-connected graphs have the same matroid if and only if they are related by a sequence of Whitney twists. Introduce a notion of Whitney twist for ribbon graphs and find an analogue of this theorem.

8. Determine the tensor product of two ribbon graphs (ribbon matroids) and find the formulas for the Bollobás-Riordan polynomial of the tensor product similar to the formulas in [Hug].

9. Introduce a notion of *adequate* (see [Th]) virtual knots and generalize Kamada's theorem [Kam] in the direction of [BM].

References

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Homology theories for links and graphs

Several years ago M. Khovanov [Kho] discovered a homology theory for links whose graded Euler characteristic is equal to the Jones polynomial. More precisely, for every link L he defined a sequence of vector spaces (abelian groups) $\mathcal{H}^{i,j}(L)$ such that

$$J_L(q) = \sum_{i,j} (-1)^i q^j \dim(\mathcal{H}^{i,j}(L)) \ .$$

There are two excellent expositions of this theory [BN, Vi]. Later on L. Helme-Guizon and Y. Rong [HGR1] found the similar theory for the chromatic polynomial of a graph. Its generalization to the Tutte polynomial can be found in [Sto].

Problems

1. Chromatic homology theory [HGR1] is based on (TQFT, *Topological Quantum Field Theory*) the algebra $A = \mathbb{R}[x]/(x^2)$. Investigate the homology theories for graphs based on the algebra $A_m = \mathbb{R}[x]/(x^m)$. Partial answers and hints may be found in [HGR2, Sto].

2. There are several theorems for the chromatic homology theory:

- For a graph Γ with n vertices and $n \geqslant 2$, $\mathcal{H}^{i,*}(\Gamma) = 0$ if $i > n-2$.
- For a graph Γ with n vertices and k connected components, $\mathcal{H}^{i,j}(\Gamma) = 0$ unless $n-k \leqslant i+j \leqslant n$.
- For a loopless connected graph Γ with n vertices only the following 0-homologies are nontrivial: $\mathcal{H}^{0,n}(\Gamma) = \mathcal{H}^{0,n-1}(\Gamma) = \mathbb{R}$ for bipartite graphs, and $\mathcal{H}^{0,n}(\Gamma) = \mathbb{R}$ for non-bipartite graphs.

Prove the following "knight move" conjecture. For a connected graph Γ with n vertices the only non-trivial homology groups $\mathcal{H}^{i,n-i}(\Gamma)$, $\mathcal{H}^{i,n-i-1}(\Gamma)$ are isomorphic in pairs: $\mathcal{H}^{i,n-i}(\Gamma) \cong \mathcal{H}^{i+1,n-i-2}(\Gamma)$ for $i \geqslant 0$ if Γ is non-bipartite, and for $i > 0$ if Γ is bipartite.
For Khovanov homology of alternating links the knight move conjecture was proved by Eun Soo Lee [Lee]. This work could be useful.

3. (M. Kidwell) Prove (or disprove) that the Whitney twist (see problem 7 of the previous project) does not change the chromatic homology of a graph.

4. Prove that chromatic homologies of a graph are determined by its chromatic polynomial. It seems this follows from the knight move conjecture of problem 2, and gives a proof of problem 3.

5. Find a natural class of graph homomorphisms that leads to a corresponding homomorphism of chromatic homologies. Do the covering maps of graphs induce such a homomorphism? Which way (is the chromatic homology a covariant or a contravariant functor)?

6. Investigate the categorification of the Tutte polynomial from [Sto]. In particular, whether it is possible to categorify the Bollobás-Riordan polynomial of ribbon graphs from previous project.

7. Another characterization of the chromatic polynomial of a graph as an Euler characteristic was found in [EH]. Find the similar characterizations for the Tutte and the Bollobás-Riordan polynomials.

8. A spanning tree model for the Khovanov homology was suggested in [CKV, Weh]. Find the similar models for chromatic, Tutte and Bollobás-Riordan homologies.

9. Clarify the situation with the Khovanov type homology theories for virtual knots. They papers [APS, Man, TT] are relevant to this.

10. Find the Khovanov homology interpretation of the results of Bae and Morton (see the paper [BM] from the previous project). Do they give an algorithm for computing the Khovanov homology groups corresponding to extreme powers of q in the Jones polinomial?

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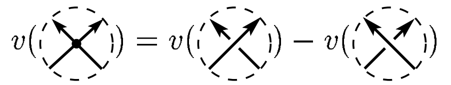
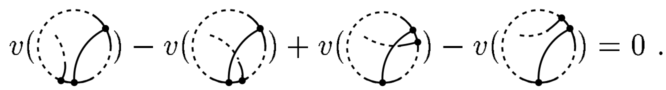
Finite type invariants of knots

The notion of *finite type* (*Vassiliev*) knot invariants appeared around 1990 in V. Vassiliev's investigations of discriminants in the (infinite-dimensional) spaces of smooth maps from one manifold into another. The idea is to extend a knot invariant v to singular knots with double points according to the rule



known as *Vassiliev's skein relation.* An invariant v is called a Vassiliev invariant of order $\leqslant n$ if its extension vanishes on all singular knots with more than n double points. The main combinatorial object underlying this theory is the a function on chord diagrams (a circle with a bunch of chords in it) satisfying the following 4-term relation. Such a function is called *weight system.*



A reader-friendly introduction to this theory see in a draft of our book [CD]. There is a general way to construct a weight system starting from a (semisimple) Lie algebra and its linear representation. The corresponding knot invariants are known as *quantum invariants.* In fact finding new weight systems was one of the motivation of the work of B. Bollobás and O. Riordan (see [BR2] in the first project). In spite of them writing (at the bottom of p.517): "Thus the space of solutions to (2) (*our 4T relation*) neither contains nor is contained in the space of weight systems ...", they do find a weight system. However this weight system was already known as a weight system associated with the Lie algebra $\mathfrak{gl}_N$ and its standard representation. Nevertheless the importance of [BR2] is that out of the $\mathfrak{gl}_N$ weight system they managed to construct a polynomial invariant (the Bollobás-Riordan polynomial) of ribbon graphs generalizing the Tutte polynomial. This leads us to our first problem.

Problems

1. Using the construction of Lie algebra weight system from [CD] for a Lie algebra and its representation construct a polynomial invariant of ribbon graphs.

2. It is known that the chromatic polynomial of the *intersection graph* (see [CD, Sec.4.3]) of a chord diagram is a weight system. This chromatic weight system is determined by Lie algebra weight systems [L]. Are there any other polynomial invariants of graphs whose values on the intersection graphs give a weight system?

3. A paper [CDL] contains an operation on chord diagrams which is analogous to mutation on knots, and which does not alter the intersection graph. Prove that any two chord diagrams with the same intersection graph are related by a sequence of these operations.

4. It is known that the universal $\mathfrak{sl}_2$ weight system [CD, Sec.7.1.3] is not changed under the operations from the previous problem. Is it true that it depends only on the intersection graph? How to compute it directly from the graph?

5. A way to describe Vassiliev invariants of a knot in terms of its Gauss diagram was found by M. Polyak and O. Viro (see [CD, Ch.13]). Find the Polyak-Viro formulas for the the coefficients of the Jones polynomial. Such formulas should work for virtual knots as well. This problem should

not be difficult because due to L. Zulli there if a formula for the entire Jones polynomial in terms of Gauss diagram [CD, Sec.13.2]

6. Construct a theory of finite type invariants of two component links which distinguishes the chords connecting different circles from the chords connecting a circle with itself.

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