1 Reduced Homologies

Definition 1.1. To define the reduced homologies $\tilde{H}^{i,j}(\Gamma)$ of a graph Γ, in each connected component K_i of Γ we select a vertex v_i and consider only those enhanced states in which v_i are colored X. d is defined on this set and is still a graded differential. We define $H^{i,j}(\Gamma)$ to be the usual i,j-th homology with respect to d, and $\tilde{H}^{i,j}(\Gamma) = H^{i,j}(\Gamma)\{-1\}$, the homologies shifted one degree down.

We will denote the reduced chain complexes by $\tilde{C}^{i,j}$ or $\tilde{C}^{i}$.

Remark 1.2. Reduced homologies are independent of the ordering on edges, since the isomorphism in [HGR, Theorem 14] works verbatim in this case.

Remark 1.3. The homology groups of the graph are tensor products of the homology groups of the connected components. Therefore, in all that follows we consider only connected graphs.

Remark 1.4. The long exact sequence of homology groups also exists for the reduced homologies, since the differential d commutes with the maps from the short exact sequence of complexes

$$0 \to \tilde{C}^{i-1}(\Gamma/e) \to \tilde{C}^{i}(\Gamma) \to \tilde{C}^{i}(\Gamma - e) \to 0,$$

in the same way as for the non-reduced chain complexes.

Remark 1.5. As in [HGR, Propositions 19, 20], the homology groups of a graph with a loop are trivial, and the homology groups of a graph with multiple edges are unchanged if the multiple edges are replaced by single edges. Hence, in all that follows, the graphs will be simple.

Remark 1.6. If e is a pendant edge of a graph Γ then $\tilde{H}^{i,j}(\Gamma) \equiv \tilde{H}^{i,j}(\Gamma/e) \otimes \mathbb{R}(q)$. The proof is exactly the same as in [HGR, Theorem 24] since it only used the long exact sequence of homologies and the fact that 1 is an identity in the algebra. Note that for a single vertex the reduced homology group is simply $\mathbb{R}$, so for a tree on n vertices the reduced homology group is $\mathbb{R}(q^{n-1})$.

2 Properties of Reduced Homologies

Proposition 2.1. $\tilde{H}^{i,j}(\Gamma) = 0$ *unless* $i + j = n - 1$*, where* n *is the number of vertices of* Γ*.*

Proof. The proof is by induction on the number of edges.

Base Case. There is only one graph with 0 edges: the one-vertex tree. The homology of a single vertex is $\mathbb{R}$.

Induction Step. If Γ is a tree, the assertion of the proposition follows from Remark 1.6. Else, let e be some edge whose removal does not disconnect Γ. The relevant portion of the long exact sequence is as follows:

$$\ldots \to \tilde{H}^{i-1,j}(\Gamma/e) \to \tilde{H}^{i,j}(\Gamma) \to \tilde{H}^{i,j}(\Gamma - e) \to \ldots$$

Since Γ/e and $\Gamma - e$ have fewer edges than Γ, by the induction hypothesis $\tilde{H}^{i-1,j}(\Gamma/e) = 0$ unless $i - 1 + j = (n-1) - 1$ and $\tilde{H}^{i,j}(\Gamma - e) = 0$ unless $i + j = n - 1$. From exactness, $H^{i,j}(\Gamma) = 0$ unless $i + j = n - 1$. □

Proposition 2.2. $\tilde{H}^{i,j}(\Gamma) = \tilde{H}^{i-1,j}(\Gamma/e) \oplus \tilde{H}^{i,j}(\Gamma - e)$.

Proof. Note that unless $i + j = n$, by Proposition 2.1 all the homologies are zero. When $i + j = n$, the relevant segment of the long exact sequence looks as follows:

$$0 = \tilde{H}^{i-1,j}(\Gamma) \to \tilde{H}^{i-1,j}(\Gamma/e) \to \tilde{H}^{i,j}(\Gamma) \to \tilde{H}^{i,j}(\Gamma - e) \to \tilde{H}^{i,j}(\Gamma/e) = 0$$

By exactness, $\tilde{H}^{i,j}(\Gamma) = \tilde{H}^{i-1,j}(\Gamma/e) \oplus \tilde{H}^{i,j}(\Gamma - e)$. □

Proposition 2.3. *Reduced homologies* $\tilde{H}^{i,j}(\Gamma)$ *are independent of the choice of special vertex.*

Proof. The proof is by induction on the number of edges.

Base Case. If Γ has no edges, then it is the one-vertex graph, and there is nothing to prove.

Induction Step. For Γ a tree, the proposition follows from remark 1.6, since the homology groups are the same regardless of vertex choice. Else, let Γ and Γ' correspond to the same graph but with different special vertices, v and v' respectively. Let e be an edge of Γ (also of Γ') whose removal does not disconnect Γ. By Proposition 2.2,

$$\tilde{H}^{i}(\Gamma) = \tilde{H}^{i-1}(\Gamma/e) \oplus \tilde{H}^{i}(\Gamma - e)$$
$$\tilde{H}^{i}(\Gamma') = \tilde{H}^{i-1}(\Gamma'/e) \oplus \tilde{H}^{i}(\Gamma' - e)$$

By inductive hypothesis, $\tilde{H}^{i-1}(\Gamma/e) \cong \tilde{H}^{i-1}(\Gamma'/e)$ and $\tilde{H}^{i}(\Gamma - e) \cong \tilde{H}^{i}(\Gamma' - e)$, since these represent the same graph but with different special vertices, the images of v and v' respectively. Hence, $\tilde{H}^{i}(\Gamma) \cong \tilde{H}^{i}(\Gamma')$. □

3 Homologies of Union

The motivation for introducing reduced homologies is the following property of the chromatic polynomial: if Γ is obtained from Γ_1 and Γ_2 by taking a vertex $v_1 \in \Gamma_1$ and a vertex $v_2 \in \Gamma_2$ and glueing them together ($\Gamma = \Gamma_1 * \Gamma_2$), then

$$\chi_\Gamma(\lambda) = \frac{\chi_{\Gamma_1}(\lambda)\,\chi_{\Gamma_2}(\lambda)}{\lambda}.$$

Introducing the reduced polynomial $\tilde{\chi}_\Gamma(\lambda) = \lambda^{-1}\chi_\Gamma(\lambda)$, we get

$$\tilde{\chi}_\Gamma(\lambda) = \tilde{\chi}_{\Gamma_1}(\lambda)\,\tilde{\chi}_{\Gamma_2}(\lambda).$$

We now establish that the reduced homologies form the categorification of the reduced chromatic polynomial and have this property of the chromatic polynomial.

Proposition 3.1. *The graded Euler characteristic of the reduced chain complex $\tilde{C}(G)$ is equal to the reduced chromatic polynomial $\tilde{\chi}_\Gamma(\lambda)$ with $\lambda = 1+q$.*

Proof. The assertion is easily shown by induction on the number of edges, since (base case) for a tree on n vertices the reduced chromatic polynomial is $\lambda^{-1}\big(\lambda(\lambda-1)^{n-1}\big) = (\lambda-1)^{n-1} = q^{n-1}$ and the reduced homology group is $\mathbb{R}(q^{n-1})$. □

Proposition 3.2. *If $\Gamma = \Gamma_1 * \Gamma_2$, then $\tilde{H}(\Gamma) = \tilde{H}(\Gamma_1) \otimes \tilde{H}(\Gamma_2)$.*

Proof. Since the homology groups are independent of the choice of special vertex, we may suppose Γ_1 and Γ_2 are joined by identifying their special vertices with each other; the resulting vertex is special in the union.

The proof is done by induction on the number of edges of Γ_2.

Base Case. If Γ_2 has no edges, then it is the single-vertex graph, so $\tilde{H}(\Gamma) = \tilde{H}(\Gamma_1)$ (since the graphs are the same), and $\tilde{H}(\Gamma_2) = \mathbb{R}$.

Inductive Step. If Γ_2 is a tree with n vertices, $\tilde{H}(\Gamma)$ is obtained from $\tilde{H}(\Gamma_1)$ via Proposition 1.6 as $\tilde{H}(\Gamma) = \tilde{H}(\Gamma_1) \otimes \mathbb{R}(q^{n-1})$. By the same proposition, $\mathbb{R}(q^{n-1})$ is the homology of Γ_2. If Γ_2 is not a tree, let e be an edge of Γ_2 whose removal doesn't disconnect Γ_2. By Proposition 2.2,

$$\tilde{H}(\Gamma_2) = \tilde{H}(\Gamma_2/e) \oplus \tilde{H}(\Gamma_2 - e),$$
$$\tilde{H}(\Gamma) = \tilde{H}(\Gamma/e) \oplus \tilde{H}(\Gamma - e).$$

By inductive hypothesis,

$$\tilde{H}(\Gamma/e) = \tilde{H}(\Gamma_1) \otimes \tilde{H}(\Gamma_2/e),$$
$$\tilde{H}(\Gamma - e) = \tilde{H}(\Gamma_1) \otimes \tilde{H}(\Gamma_2 - e).$$

Taking the direct sum,

$$\tilde{H}(\Gamma) = \tilde{H}(\Gamma_1) \otimes \big(\tilde{H}(\Gamma_2/e) \oplus \tilde{H}(\Gamma_2 - e)\big) = \tilde{H}(\Gamma_1) \otimes \tilde{H}(\Gamma_2).$$

□

4 Matroid Type

A *Whitney twist* on a graph Γ can be defined as follows [Wh, Hug]. Let Γ_1 and Γ_2 be two graphs. Pick edges $e_1 \in \Gamma_1$ and $e_2 \in \Gamma_2$. Construct a new graph by gluing the edges $e_1 \in \Gamma_1$ and $e_2 \in \Gamma_2$ together (with their endpoints) and then removing the resulting single edge from the graph. In general this can be done in two ways depending on how we glue e_1 with e_2. If one of them is Γ then the other is a Whitney twist of Γ. Whitney proved that two 2-connected graphs have the same matroid type iff one can be obtained from the other by a sequence of Whitney twists.

We show that the reduced homology sequence of a graph is invariant under the Whitney twist. From this we derive that the reduced homology sequence is an invariant of the matroid type of the graph.

Proposition 4.1. *If Γ and Γ' are related by a Whitney twist, $\tilde{H}(\Gamma) \cong \tilde{H}(\Gamma')$.*

Proof. Let G and G' be obtained by joining Γ_1 and Γ_2 along e. By Proposition 2.2,

$$\tilde{H}^i(G) = \tilde{H}^{i-1}(G/e) \oplus \tilde{H}^i(G-e)$$

and similarly for G'. Set $\Gamma = G - e$ and $\Gamma' = G' - e$. We are interested in proving $\tilde{H}^i(G-e) \cong \tilde{H}^i(G'-e)$, but it suffices to prove the isomorphism for the other two homology sequences corresponding to G and G'.

Note that $G/e = (\Gamma_1/e) * (\Gamma_2/e) = G'/e$; hence, the homologies in both cases are just the tensor product of the homologies of Γ_1/e and Γ_2/e. To show the isomorphism of the homology groups of G and G', we induct on the number of edges of Γ_2.

Base Case. Γ_2 cannot have less than one edge, since we have to glue Γ_1 and Γ_2 together along an edge. If Γ_2 has exactly one edge, then $\Gamma = \Gamma_1$ regardless of the orientation of the glueing of this edge.

Inductive Step. If Γ_2 is a tree on n vertices, then Γ is obtained from Γ_1 by adding two subtrees of Γ_2 with a total of $n-2$ edges. By Proposition 1.6, $\tilde{H}(\Gamma) = \tilde{H}(\Gamma_1) \otimes \mathbb{R}(q^{n-2})$ regardless of the orientation of e.

If Γ_2 is not a tree, let $e' \neq e \in \Gamma_2$ be such that its removal does not disconnect Γ_2. Then $\tilde{H}^i(G) = \tilde{H}^{i-1}(G/e') \oplus \tilde{H}^i(G-e')$, where G/e' and $G-e'$ are obtained by gluing Γ_2/e' and $\Gamma_2 - e'$ respectively to Γ_1 along e. Similarly, $\tilde{H}^i(G') = \tilde{H}^{i-1}(G'/e') \oplus \tilde{H}^i(G'-e')$. By the inductive assumption, $\tilde{H}^{i-1}(G/e') \cong \tilde{H}^{i-1}(G'/e')$ and $\tilde{H}^i(G-e') \cong \tilde{H}^i(G'-e')$, and hence $\tilde{H}^i(G) \cong \tilde{H}^i(G')$. □

Proposition 4.2. *The reduced homology sequence is an invariant of the matroid type of the graph.*

Proof. The proof is by induction on the number of edges. There is only one graph with no edges, so the base of induction is vacuously true.

Inductive Step. If Γ is 2-connected, we are done by Proposition 4.1. If Γ is not 2-connected, the removal of some vertex $v \in \Gamma$ breaks Γ up into two connected components, G_1 and G_2. Adding v back into G_1 and G_2 we get subgraphs Γ_1 and Γ_2 of Γ, where $\Gamma = \Gamma_1 * \Gamma_2$ (the vertex being v). Then the matroid type of Γ is the same as of the disjoint union of Γ_1 and Γ_2. On the other hand, $\tilde{H}(\Gamma) = \tilde{H}(\Gamma_1) \otimes \tilde{H}(\Gamma_2)$, which is also equal to the homology sequence of the disjoint union of Γ_1 and Γ_2. □

5 Relationship to Non-Reduced Homologies and the Chromatic Polynomial

Here we derive the relationship between the reduced homologies of graphs and the main diagonal of the usual homologies. Then we describe the Poincaré polynomial for the reduced homologies in terms of the reduced chromatic polynomial.

Proposition 5.1. *For C_n a cycle with n vertices, $\tilde{H}^i(C_n) = \mathbb{R}(q^{n-i-1})$ when $0 \leq i \leq n-2$; outside this range, $\tilde{H}^n(C_n) = 0$.*

Proof. We induct on n.

Base Case. If $n = 1$, we have a loop, whose homologies are zero. If $n = 2$, the graph has two vertices and two edges, so its homology sequence is $H^0(C_2) = \mathbb{R}(q)$ (and zero for the first and greater homology groups).

Inductive Step. If $i = 0$, the zeroth homology group of any connected n-vertex graph is $\mathbb{R}(q^{n-1})$. For $1 \leq i$, $\tilde{H}^i(C_n) \cong \tilde{H}^{i-1}(C_n/e) \oplus \tilde{H}^i(C_n - e)$. Now $C_n/e = C_{n-1}$, and $C_n - e = T_n$, a tree on n vertices. By inductive assumption, $\tilde{H}^{i-1}(C_{n-1}) = \mathbb{R}(q^{n-i-1})$ whenever $0 \leq i-1 \leq n-3$, i.e. $1 \leq i \leq n-2$; outside this range, $\tilde{H}^{i-1}(C_{n-1}) = 0$. $\tilde{H}^i(T_n) = \mathbb{R}(q^{n-1})$ if $i = 0$, and 0 otherwise. Adding the two, we get the statement of the proposition. □

Proposition 5.2. *Let $R^n_\Gamma(t,q)$ be the Poincaré polynomial for the $i+j=n$ diagonal of the usual homologies, and similarly for $R^{n-1}_\Gamma(t,q)$. The Poincaré polynomial for the reduced homologies is*

$$\tilde{R}_\Gamma(t,q) = \begin{cases} \frac{1}{q}\left(R^n_\Gamma(t,q)\left(1+\frac{t}{q}\right) - tq^{n-1}\right) & = \frac{1}{q}R^n_\Gamma(t,q) + R^{n-1}_\Gamma(t,q) - q^{n-1}, \\ \quad \textit{if } \Gamma \textit{ is bipartite} & \\ \frac{1}{q}R^n_\Gamma(t,q)\left(1+\frac{t}{q}\right) & = \frac{1}{q}R^n_\Gamma(t,q) + R^{n-1}_\Gamma(t,q), \\ \quad \textit{otherwise} & \end{cases}$$

Proof. The second set of equalities is a direct consequence of [CH, Theorem 4.2]. We prove the first set, by induction on the number of edges.

First, we examine the cases of a tree and of a single odd-length-cycle graph. If Γ is a tree, $\tilde{R}_\Gamma(t,q) = q^{n-1}$. On the other hand, the non-reduced homologies of Γ are $H(\Gamma) = \mathbb{R}(q^n) \oplus \mathbb{R}(q^{n-1})$ (see [HGR, Example 28]).

Thus, $R^n_\Gamma(t,q) = q^n$ and $R^{n-1}_\Gamma(t,q) = q^{n-1}$. We observe

$$q^{n-1} = \frac{1}{q}\big(q^n\left(1+\frac{t}{q}\right) - tq^{n-1}\big),$$

as expected since trees are bipartite.

If Γ is a single cycle of length n (odd), then

$$\tilde{R}_\Gamma(t,q) = q^{n-1} + q^{n-2}t + \ldots + qt^{n-2}$$

The non-reduced Poincaré polynomial, from [HGR, Example 29], is

$$R^n_\Gamma(t,q) = q^n + q^{n-2}t^2 + \ldots + q^3t^{n-3}$$

The proposition follows by explicit computation.

Note that if $\Gamma = \Gamma_1 * \Gamma_2$ where Γ_2 is a tree on n vertices, then both the reduced and the non-reduced homologies of Γ are computed by taking the respective homologies of Γ_1 and multiplying by $\mathbb{R}(q^{n-1})$. Since adding a tree in this fashion preserves the bipartite or non-bipartite property, it also preserves the equality of polynomials above.

Now we proceed to the proper induction step. If Γ is not bipartite and contains more than one cycle (the one-cycle case was discussed above), then Γ contains some edge e whose removal does not disconnect Γ and whose contraction does not make Γ bipartite. (Pick the smallest odd cycle C of Γ. We know that Γ has some other cycle, $C' \not\subseteq C$. Pick an edge $e \in C' - C$.) Note that both for the regular and the reduced Poincaré polynomials we have the equation

$$R_\Gamma(t,q) = R_{\Gamma/e}(t,q) + R_{\Gamma-e}(t,q).$$

By construction, Γ/e and $\Gamma - e$ are both not bipartite. By inductive assumption, we may assume that the terms on the right-hand side satisfy the correct relations. Then

$$\tilde{R}_\Gamma(t,q) = \frac{1}{q}R^n_{\Gamma/e}(t,q)\left(1+\frac{t}{q}\right) + \frac{1}{q}R^n_{\Gamma-e}(t,q)\left(1+\frac{t}{q}\right) = \frac{1}{q}R^n_\Gamma(t,q)\left(1+\frac{t}{q}\right)$$

If Γ is bipartite and contains a cycle, we take an edge e contained in some (even) cycle. Then $\Gamma - e$ will still be bipartite, but Γ/e will not be bipartite. Therefore,

$$\begin{aligned}\tilde{R}_\Gamma(t,q) &= \frac{1}{q}R^n_{\Gamma/e}(t,q)\left(1+\frac{t}{q}\right) + \frac{1}{q}\big(R^n_{\Gamma-e}(t,q)\left(1+\frac{t}{q}\right) - tq^{n-1}\big) \\ &= \frac{1}{q}\big(R^n_\Gamma(t,q)\left(1+\frac{t}{q}\right) - tq^{n-1}\big)\end{aligned}$$

□

[CH, Theorem 4.2] derives $R^n_\Gamma(-1,q)$ from $\chi_\Gamma(q)$. The corresponding result for the reduced homologies is much simpler. The chromatic polynomial is the specification of the Poincaré polynomial at $t=-1$. The Poincaré polynomial for the reduced homologies is homogeneous of degree $n-1$ by Proposition 2.1. Thus, $\tilde{R}_\Gamma(t,q)$ is completely determined by $\tilde{R}_\Gamma(-1,q)$. Specifically, $\tilde{R}_\Gamma(t,q) = q^{n-1}\chi_\Gamma\left(-\frac{t}{q}\right)$.

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