Bollobás-Riordan Polynomial of a Ribbon Graph

Abstract

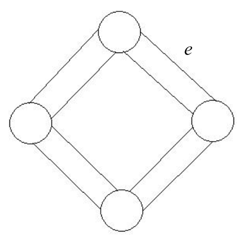
This paper is a summary of a talk given by Timothy All at the Ohio State University, Summer quarter 2005 in the VIGRE working group "Knot Theory and Combinatorics". We shall define *ribbon graphs* and introduce the four polynomial invariant according to [1]. We shall then summarize the main results of [1], that is we shall prove that the aforementioned polynomial $R(\mathbf{G})$ satisfies the relation $R(\mathbf{G}) = R(\mathbf{G}/e)+R(\mathbf{G}-e)$.

1 Ribbon Graphs

Ribbon graphs can informally be defined as disks attached to each other by thin strips. Keeping this intuitive idea of ribbon graphs in mind, formally we have that a ribbon graph $\mathbf{G}$ is a surface S with boundary with two finite sets of disks: a set $V(\mathbf{G})$ of vertices and a set $E(\mathbf{G})$ of edges. These sets are subject to the following restrictions:

- disks and ribbons intersect by disjoint line segments
- each such line segment lies on the boundary of precisely one vertex and one edge
- every edge contains exactly two such line segments

Example A ribbon graph $\mathbf{G}$ with edge e.



Now, let us define some graph parameters for a ribbon graph $\mathbf{G}$. Let

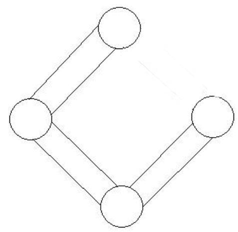
$$v(\mathbf{G}) := \|V(\mathbf{G})\|, \text{the number of vertices of } \mathbf{G} \quad (1)$$
$$e(\mathbf{G}) := \|E(\mathbf{G})\|, \text{the number of edges of } \mathbf{G} \quad (2)$$
$$k(\mathbf{G}) := \text{the number of connected components of } \mathbf{G} \quad (3)$$
$$r(\mathbf{G}) := v(\mathbf{G}) - k(\mathbf{G}), \text{the rank of } \mathbf{G} \quad (4)$$
$$n(\mathbf{G}) := e(\mathbf{G}) - r(\mathbf{G}), \text{the nullity of } \mathbf{G} \quad (5)$$
$$bc(\mathbf{G}) := \text{the number of components of the boundary of } \mathbf{G} \quad (6)$$
$$t(\mathbf{G}) := \text{the orientablility of the surface} \mathbf{G} \quad (7)$$

We say that $t(\mathbf{G}) = 0$ if $\mathbf{G}$ is orientable and $t(\mathbf{G}) = 1$ if otherwise. Note: $k(\mathbf{G})$ corresponds to the connectedness of $\mathbf{G}$ as a graph whereas $bc(\mathbf{G})$ corresponds to the connectedness of the boundary of $\mathbf{G}$ as a geometric ribbon graph or surface. Moreover, we say that an edge is a *loop* if both ends of the edge are adjacent to the same vertice, and an edge is called a *bridge* if its removal would disconnect a component of $\mathbf{G}$, i.e. if it alters the value of $k(\mathbf{G})$. If e is neither a loop nor a bridge, then it is said to be ordinary.

2 Contracting and Deleting Edges, and Combining Ribbon Graphs

There are four operations that can be performed on a ribbon graph and they affect the graph parameters in varying ways. The first such operation we shall consider will be deleting or removing an edge. To delete an edge from a ribbon graph we simply remove the edge e from the set $E(\mathbf{G})$ and denote the resulting graph by $\mathbf{G} - e$.

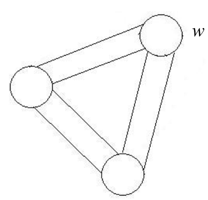
Example Let $\mathbf{G}$ be the ribbon graph of the previous example with edge e. Then $\mathbf{G} - e$ is the ribbon graph:



If e is an ordinary edge or a loop, then deletion of e only affects $e(\mathbf{G})$ such that $e(\mathbf{G} - e) = e(\mathbf{G}) - 1$. All other parameters remain unchanged. If e is a bridge, then in addition we have that $k(\mathbf{G} - e) = k(\mathbf{G}) + 1$.

For an edge e that is not a loop, we may also contract e in $\mathbf{G}$. Let e be adjacent to the vertices v_1 and v_2. To contract e, we delete e, v_1, and v_2 and introduce the new vertice $e \cup v_1 \cup v_2$. We denote the resulting graph by $\mathbf{G}/e$.

Example Using our familiar example where $e \cup v_1 \cup v_2 = w$, we see that $\mathbf{G}/e$ is the graph:

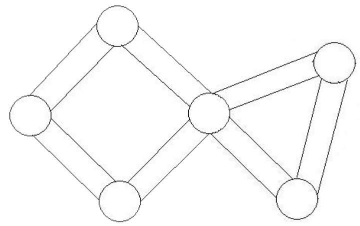


For contraction of an edge e, $k(\mathbf{G}), n(\mathbf{G}), bc(\mathbf{G})$, and $t(\mathbf{G})$ remain unchanged while $v(\mathbf{G})$ and $r(\mathbf{G})$ decrease by one.

Now, we shall consider how we may combine two ribbon graphs. The first and most obvious way to join two ribbon graphs $\mathbf{G}_1$ and $\mathbf{G}_2$ is to take the disjoint union. We denote this combination by $\mathbf{G}_1 \uplus \mathbf{G}_2$. The parameters $v(\mathbf{G}), e(\mathbf{G}), k(\mathbf{G}), r(\mathbf{G}), n(\mathbf{G})$, and $bc(\mathbf{G})$ are all additive with respect to disjoint unions.

The other method of combining ribbon graphs is to take the connected sum or in other words to create a one-point join. To do this we choose any vertice v_1 in $\mathbf{G}_1$ and any vertice v_2 in $\mathbf{G}_2$, then we pin the two ribbon graphs together at these vertices by identifying a segment of the boundary of v_1 with a segment of the boundary of v_2. We denote this combination by $\mathbf{G}_1 \cdot \mathbf{G}_2$.

Example Refering to our previous examples, this would be one way to form the connected sum of $\mathbf{G}$ and $\mathbf{G}/e$:



Note that $e(\mathbf{G}), r(\mathbf{G})$, and $n(\mathbf{G})$ are additive with respect to one-point joins, while $v(\mathbf{G}), k(\mathbf{G})$, and $bc(\mathbf{G})$ lose one, that is $k(\mathbf{G}_1 \cdot \mathbf{G}_2) = k(\mathbf{G}_1) + k(\mathbf{G}_2) - 1, etc.$

We need only the notion of a spanning subgraph before we define the ribbon graph polynomial. A spanning subgraph of a ribbon graph $\mathbf{G}$ is a subgraph of $\mathbf{G}$ that includes all the vertices of $\mathbf{G}$ but only a subset of the edges. So, there are naturally $2^{e(\mathbf{G})}$ spanning subgraphs of any ribbon graph $\mathbf{G}$.

3 The Ribbon Graph Polynomial

The ribbon graph polynomial is similar to the Tutte polynomial but with some extra topological information.

Definition The ribbon graph polynomial $R(\mathbf{G})$ is defined as follows:

$$R(\mathbf{G}) \quad = \quad \sum_{\mathbf{H} \subset \mathbf{G}} (X-1)^{r(\mathbf{G})-r(\mathbf{H})} Y^{n(\mathbf{H})} Z^{k(\mathbf{H})-bc(\mathbf{H})+n(\mathbf{H})} W^{t(\mathbf{H})} \tag{8}$$

where the sum is taken over all spanning subgraphs $\mathbf{H}$, and $R(\mathbf{G}) \in \mathbf{Z}[X,Y,Z,W]/(W^2 - W)$.

Note that $v(\mathbf{H}) = v(\mathbf{G}$, so $r(\mathbf{G}) - r(\mathbf{H}) = k(\mathbf{H}) - k(\mathbf{G})$. Using this equality we can rewrite (8) as follows:

$$R(\mathbf{G}) \quad = \quad (X-1)^{-k(\mathbf{G})} \sum_{\mathbf{H} \subset \mathbf{G}} M(\mathbf{H}) \tag{9}$$

where

$$M(\mathbf{H}) \quad = \quad (X-1)^{k(\mathbf{H})} Y^{n(\mathbf{H})} Z^{k(\mathbf{H})-bc(\mathbf{H})+n(\mathbf{H})} W^{t(\mathbf{H})} \tag{10}$$

Also note that $M(\mathbf{H}_1 \uplus \mathbf{H}_2) = M(\mathbf{H}_1)M(\mathbf{H}_2)$ since all the parameters in the exponents of $M(\mathbf{H})$ are additive with respect to disjoint unions. What's more, since $k(\mathbf{H}_1 \cdot \mathbf{H}_2) = k(\mathbf{H}_1) + k(\mathbf{H}_2) - 1$ and $bc(\mathbf{H}_1 \cdot \mathbf{H}_2) = bc(\mathbf{H}_1) + bc(\mathbf{H}_2) - 1$ we have that $M(\mathbf{H}_1)M(\mathbf{H}_2) = (X-1)M(\mathbf{H}_1 \cdot \mathbf{H}_2)$. As $\mathbf{H}_1$ and $\mathbf{H}_2$ run over the spanning subgraphs of $\mathbf{G}_1$ and $\mathbf{G}_2$ respectively, we have that $\mathbf{H}_1 \uplus \mathbf{H}_2$ and $\mathbf{H}_1 \cdot \mathbf{H}_2$ run over the spanning subgraphs of $\mathbf{G}_1 \uplus \mathbf{G}_2$ and $\mathbf{G}_1 \cdot \mathbf{G}_2$ respectively. And so we find that

$$\begin{aligned} R(\mathbf{G}_1 \uplus \mathbf{G}_2) &= (X-1)^{k(\mathbf{G}_1 \uplus \mathbf{G}_2)} \sum_{\mathbf{H}_1 \subset \mathbf{G}_1, \mathbf{H}_2 \subset \mathbf{G}_2} M(\mathbf{H}_1)M(\mathbf{H}_2) \\ &= R(\mathbf{G}_1)R(\mathbf{G}_2) \end{aligned}$$

and

$$\begin{aligned} R(\mathbf{G}_1 \cdot \mathbf{G}_2) &= (X-1)^{k(\mathbf{G}_1 \cdot \mathbf{G}_2)} \sum_{\mathbf{H}_1 \subset \mathbf{G}_1, \mathbf{H}_2 \subset \mathbf{G}_2} M(\mathbf{H}_1)M(\mathbf{H}_2)(X-1)^{-1} \\ &= (X-1)^{-k(\mathbf{G}_1)-k(\mathbf{G}_2)+1-1} \sum_{\mathbf{H}_1 \subset \mathbf{G}_1, \mathbf{H}_2 \subset \mathbf{G}_2} M(\mathbf{H}_1)M(\mathbf{H}_2) \\ &= R(\mathbf{G}_1)R(\mathbf{G}_2) \end{aligned}$$

4 Main Theorem

Theorem 4.1 *Let* $\mathbf{G}$ *be any ribbon graph, then*

$$R(\mathbf{G}) = R(\mathbf{G}/e) + R(\mathbf{G}-e) \tag{11}$$

for every ordinary edge e *of* $\mathbf{G}$*, and*

$$R(\mathbf{G}) = XR(\mathbf{G}/e) \tag{12}$$

for every bridge e *of* $\mathbf{G}$*.*

Proof The subgraphs of $\mathbf{G}$ not containing e are all the subgraphs of $\mathbf{G}-e$. Now, let $\varphi : \mathbf{H} \to \mathbf{H}/e$ such that φ is a bijection between the subgraphs of $\mathbf{G}$ containing e to the subgraphs of $\mathbf{G}/e$. Since the parameters in the exponents of $M(\mathbf{H})$ are all unchanged by the contraction of an ordinary edge e, we have that $M(\mathbf{H}) = M(\mathbf{H}/e)$. Now, suppose e is an ordinary edge, so that $k(\mathbf{G}) = k(\mathbf{G}-e) = k(\mathbf{G}/e)$, and we see that

$$\begin{aligned} R(\mathbf{G}) &= (X-1)^{-k(\mathbf{G})} \sum_{\mathbf{H}\subset\mathbf{G}, e\notin E(\mathbf{G})} M(\mathbf{H}) + (X-1)^{-k(\mathbf{G})} \sum_{\mathbf{H}\subset\mathbf{G}, e\in E(\mathbf{G})} M(\mathbf{H}) \\ &= (X-1)^{-k(\mathbf{G}-e)} \sum_{\mathbf{H}\subset\mathbf{G}-e} M(\mathbf{H}) + (X-1)^{-k(\mathbf{G}/e)} \sum_{\mathbf{H}\subset\mathbf{G}/e} M(\mathbf{H}) \\ &= R(\mathbf{G}-e) + R(\mathbf{G}/e) \end{aligned}$$

proving (11). Now suppose e is a bridge in $\mathbf{G}$ so that $k(\mathbf{G}-e) = k(\mathbf{G}) + 1$, and we see that

$$\begin{aligned} R(\mathbf{G}) &= (X-1)^{-k(\mathbf{G})} \sum_{\mathbf{H}\subset\mathbf{G}-e} M(\mathbf{H}) + (X-1)^{-k(\mathbf{G}/e)} \sum_{\mathbf{H}\subset\mathbf{G}/e} M(\mathbf{H}) \\ &= (X-1)^{-k(\mathbf{G}-e)+1} \sum_{\mathbf{H}\subset\mathbf{G}-e} M(\mathbf{H}) + R(\mathbf{G}/e) \\ &= (X-1)R(\mathbf{G}-e) + R(\mathbf{G}/e) \end{aligned}$$

But since e is a bridge in $\mathbf{G}$, we can view $\mathbf{G}-e$ as the disjoint union of two graphs $\mathbf{G}_1$ and $\mathbf{G}_2$, that is $\mathbf{G}-e = \mathbf{G}_1 \uplus \mathbf{G}_2$. And similarly we may view $\mathbf{G}/e$ as the connected sum of $\mathbf{G}_1$ and $\mathbf{G}_2$, i.e. $\mathbf{G}/e = \mathbf{G}_1 \cdot \mathbf{G}_2$. From what has gone on before, we know that $R(\mathbf{G}-e) = R(\mathbf{G}_1 \uplus \mathbf{G}_2) = R(\mathbf{G}_1)R(\mathbf{G}_2) = R(\mathbf{G}_1 \cdot \mathbf{G}_2) = R(\mathbf{G}/e)$. And so we have that $R(\mathbf{G}) = XR(\mathbf{G}/e)$ proving (12).

References

[1] B. Bollobás, O. Riordan, *A Polynomial of Graphs on Surfaces*, Math. Ann. **323**(1) (2002), 81-96.