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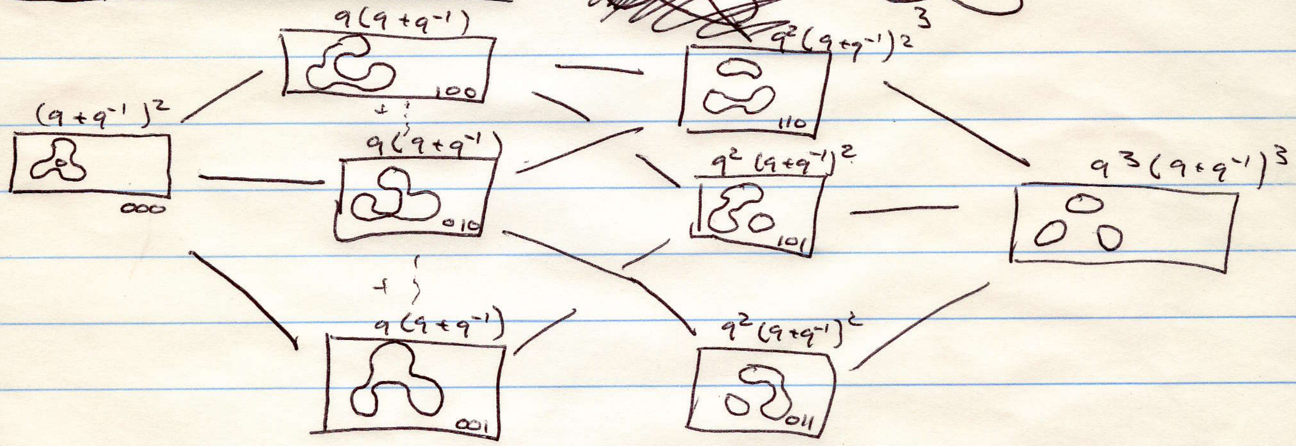
Khovanov Homology of Knots

Smoothings

Given $\times$, $\asymp$ is the 0 smoothing, and
$)($ is the 1 smoothing.

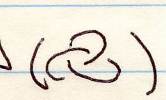
If χ is the set of all crossings of L, a link, then $\alpha \in \{0,1\}^{\chi}$ corresponds to a "complete smoothing" S_α of L, a state. The Jones Polynomial is
$$(-1)^{n_-} q^{n_+ - 2n_-} \Big(\sum_{\alpha \in \{0,1\}} (-1)^{r(\alpha)} q^{r(\alpha)} (q+q^{-1})^{k(\alpha)} \Big),$$
where $r(\alpha)$ is the height, i.e. # of 1-smoothings in S_α, and $k(\alpha)$ is the number of cycles in S_α, where $n = |\chi|$ and n_+ = # of right hand crossings, n_- = # of left hand crossings.

Example: The Trefoil



$$(q+q^{-1})^2 - 3q(q+q^{-1}) + 3q^2(q+q^{-1})^2 - q^2(q+q^{-1})^3$$

$$= q^{-2} + 1 + q^2 - q^6 \xrightarrow[(n_+,\, n_-) = (3,0)]{(-1)^{n_-} q^{n_+ - 2n_-}} q + q^3 + q^5 - q^9 \xrightarrow{(q+q^{-1})^{-1}}$$

$J($  $) = q^2 + q^6 - q^8.$

Aside : Euler characteristic and (singular) homology.

Euler characteristic defined as: $\chi(K) = \sum (-1)^p \operatorname{rank}(C_p(K))$

Theorem: $\chi(K) = \sum (-1)^p \operatorname{rank} H_p(K)$ (Munkres, Thm 22.2)

The relationship between Euler characteristic and (singular) homology is analagous to Jones polynomial and Khovanov Homology.

Suppose $W = \bigoplus_m W_m$ is a graded vector space.

Definition (graded dimension) $\operatorname{qdim} W := \sum_m q^m \dim W_m$.

Definition (degree shift) $\cdot\{l\}$ is the degree shift operation. Suppose $W = \bigoplus_m W_m$. Then $W\{l\}_m := W_{m-l}$.

Example $\operatorname{qdim} W\{l\} = q^l \cdot \operatorname{qdim} W$.

Definition (height shift) Suppose $\bar{C}$ is a chain complex: $\cdots \longrightarrow \bar{C}^r \xrightarrow{d^r} \bar{C}^{r+1} \xrightarrow{d^{r+1}} \cdots$ where $\bar{C}^r$ are (graded) vector spaces. r is the height of $\bar{C}^r$. Let $C = \bar{C}[s]$. Then $C^r = \bar{C}^{r-s}$ and the differentials shift similarly.

Now, I'll construct the chain complex for the Khovanov homology.

Let V be the graded vector space with two basis elements: v_+ and v_- whose degrees are $+1$, -1 resp.

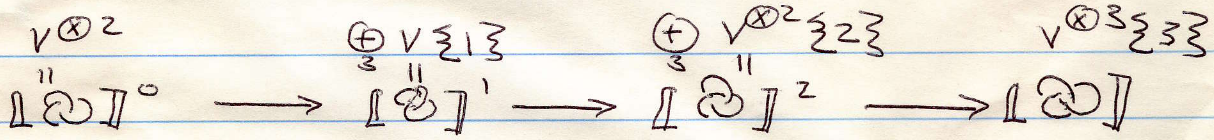
To each $\alpha \in \{0,1\}^{\mathcal{X}}$ associate the graded vector space $V_\alpha(L) := V^{\otimes k}\{r\}$ $\quad$ ($k = \#$ of cycles, $r = |\alpha| = \sum_i \alpha_i$)

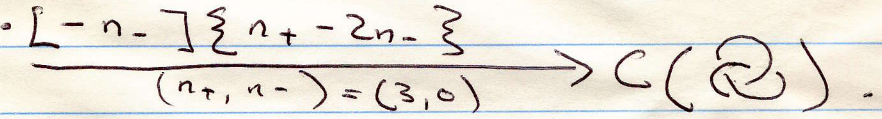
<u>Definition</u> The chain group $[\![L]\!]^r$ $(0 \le r \le n)$ is the direct sum of all vector spaces at height r: $[\![L]\!]^r := \bigoplus_{\alpha : r = |\alpha|} V_\alpha(L)$.

The graded object is

$$C(L) := [\![L]\!][-n_-]\{n_+ - 2n_-\}.$$

For the trefoil we have





<u>Check:</u> $\chi(C(L)) = \sum_r (-1)^r q \dim [\![L]\!]^r = \hat{J}(L)$

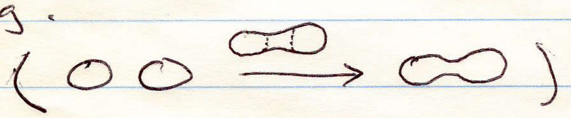
We can label the edges of $\{0,1\}^{\chi}$ by sequences in $\{0,1,\star\}^{\chi}$ where only one $\star$ appears.
The tail of the edge is the vertex w/ sequence with $\star \longmapsto 0$ and the head is the vertex w/ sequence $\star \longmapsto 1$.

i.e. $(00\star) : 000 \longrightarrow 001$.

Call the edge (or rather map) ξ and $d^r = \sum_{|\xi|=r} (-1)^{\xi} d_{\xi}$. The sign of $(-1)^{\xi}$ will be determined by first choosing them all to be positive. This makes the map commutative. Then making a subset (-1) will make it anti-commutative — hence $d^{r+1} \circ d^r = 0$.

<u>defining d_ξ:</u>

Now, ξ must be defined to make the map commute. By observation, only changing one crossing will do one of two things to the smoothing.

(i)  or $V \otimes V \xrightarrow{m} V$

(ii)  or $V \xrightarrow{\Delta} V \otimes V$

$$m : \begin{cases} V_+ \otimes V_- \longmapsto V_- & V_+ \otimes V_+ \longrightarrow V_+ \\ V_- \otimes V_+ \longmapsto V_- & V_- \otimes V_- \longrightarrow 0 \end{cases}$$

$$\Delta : \begin{cases} V_+ \longmapsto V_+ \otimes V_- + V_- \otimes V_+ \\ V_- \longmapsto V_- \otimes V_- \end{cases}$$