Philip Kilarowski: Homology of virtual knots

Classical and virtual knot diagrams

A knot diagram is a 4-valent graph w/ structure @ nodes/vertices

Classical: under- or over- crossings

A virtual knot has some crossing which are neither under nor over;

it's like they are not even there

A non-planar graph acquires virtual crossings, as the graph crosses over itself at points that are not vertices.

"Virtual crossings are not real crossings"

Classical knot $\subseteq S^2 \times I$ - plane or sphere w/o crossing itself

Virtual knot $\subseteq (\#_n T^2) \times I$ - " " surface of genus n "

Diagram w/crossings is the projection onto the plane (shadow)

Give examples of classical (trefoil) & virtual knots - show torus knot

draw K_5

Gauss codes

A sequence of labels for the crossings repeated twice

"Walk through" knot diagram

Number "actual" crossings & label under/over in the order they are traversed

Examples: trefoil, torus knot, link, $K_5 \Rightarrow$ 01 02 04 05 U2 03 05 01 03 0

01 U2 03   01- U2+   I

U1 O2 U3   U1+ O2-   partition - all others are single component

Can we tell if a graph is planar from the Gauss code?

A single component Gauss code is evenly interlaced if there is an even # of labels in between 2 appearances of any label

trefoil & K_5 Yes, "torus knot" No

Planar $\not\Rightarrow$ evenly interlaced

prove by Jordan curve thm.

Gauss code gives knot up to orientation, so we put signs on the crossing

(+) iff over-crossing can be turned through smaller angle to coincide w/ direction of under crossing

Planar graph has all positive crossings

(-) sign forces graph to have virtual crossing for change in orientation

Thm. Planar Gauss code & standard sign sequence $\Rightarrow$ Classical

Get code for Jordan curve: $\underline{123}123 \to \underline{132}123$ is doubly paired (diagram)

Planar $\to$ doubly paired $g \Rightarrow g^+$

g planar, g^* dually paired ⇒ g is the Gauss code of a planar shadow

- Checkerboard colorings

  Thicken knots $(\# T^2) \times I$

  Trace path on 1 side — step on inside/outside "ramp" @ classical crossing

  "drive through" virtual crossing

  Try to color path obtained this way in "checkerboard" pattern

  Always possible for classical link

  Possible for K_5 but not "torus knot" (demonstrated on torus)

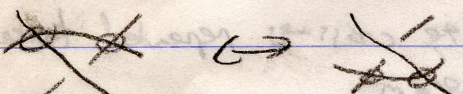
- Atoms (M, Γ)

  cpt. 2-manifold w/o boundary w/ 4-valent graph, colored as a checkerboard — more useful for dotted knots

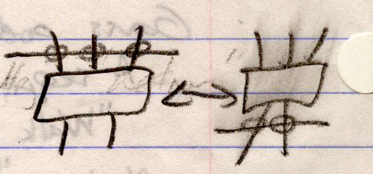
  virtual link defined by atom up to Reidemeister moves

- Reidemeister moves

  We have the same moves as we expect plus:



and a "detour"



3rd Reidemeister move preserves Gauss code

- Jones polynomial of virtual knots

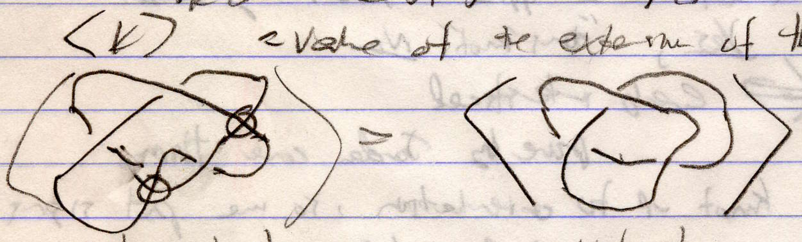
  still $d = -A^2 - A^{-2}$; $\langle K \rangle = A \langle K_a \rangle + A^{-1} \langle K_b \rangle$

  Use bracket to demonstrate invariance

  writhe $w(K) =$ sum of crossing signs (again)

  $f_K(A) = (-A^3)^{-w(K)} \langle K \rangle / N$

  $\langle K \rangle$ = value of the extension of the bracket to knots in $\#T^2 \times I$ / $S_g \times I$



virtual knot unknot

To put on board:

$m: V\otimes V \to V$

$v_+ \otimes v_- \mapsto v_-$

$v_+ \otimes v_+ \mapsto v_+$

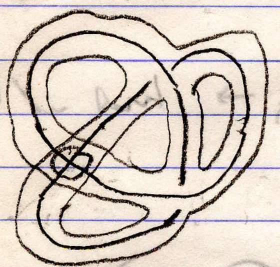
$v_- \otimes v_+ \mapsto v_-$

$v_- \otimes v_- \mapsto 0$

$\Delta: V \to V\otimes V$

$v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+$

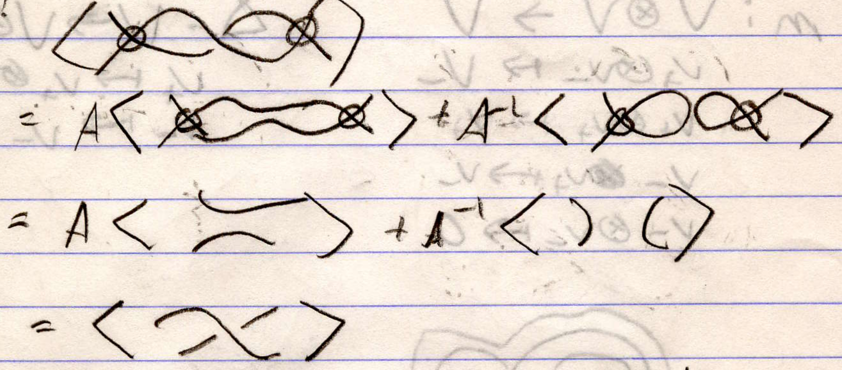
$v_- \mapsto v_- \otimes v_-$



put virtual link
@ hole in board

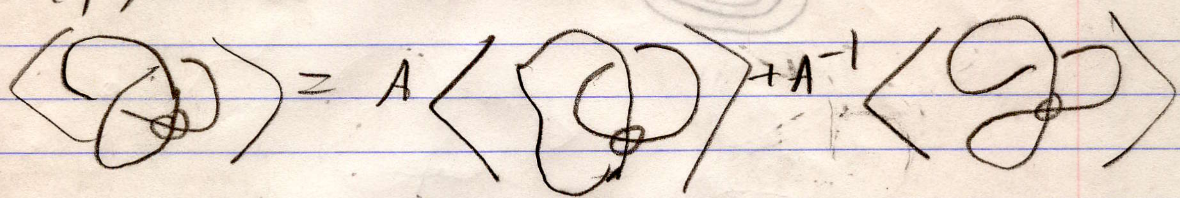
Reidemeister moves

Forbidden
Quandle!



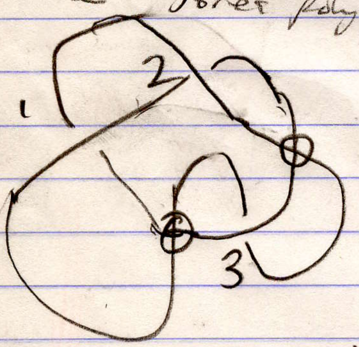
Start demonstration of $K_4 \to$ knot w/ 1 virtual crossing

Jones polynomial of "torus knot"



$A^2 + 1 - A^{-4}$

Knot with same Jones polynomial as Un-knot



Jones poly $= -A^3$ (trivial)

O1+ U2+ O3− U1+ O2+ U3−

- ~~Kh~~ Khovanov homology of virtual knots

Resolutions are $\{0,1\}^n$

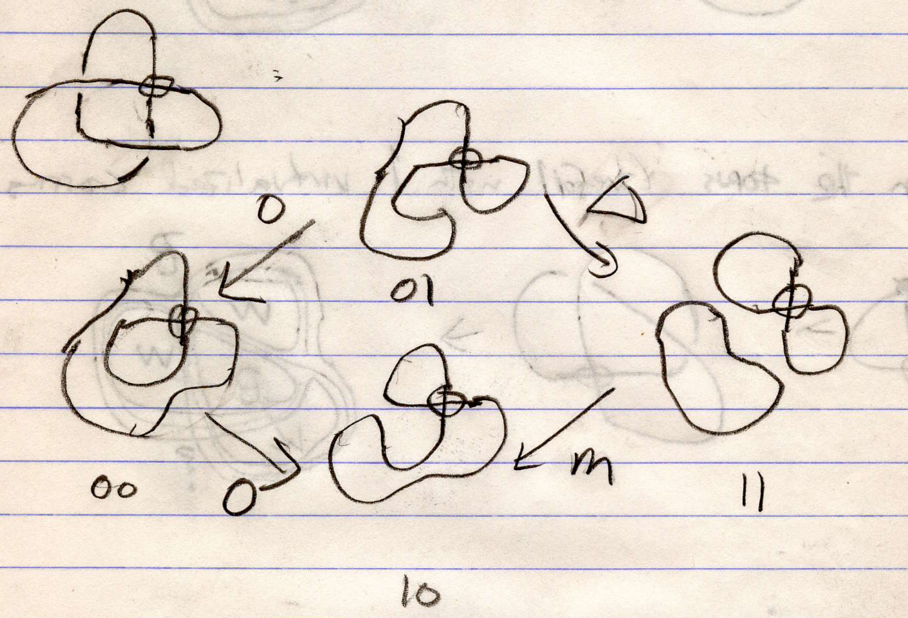
$V = \langle v_+, v_- \rangle$ are $\mathbb{Z}_2$

grading of $v_+, v_- = \pm 1$

$\gamma(s) = \#$ of circles

vertices are $\Rightarrow V^{\otimes \gamma(s)} \sum_{i=0}^{n} a_i$

$q \dim V = q + q^{-1}$



"1-1 resolution" 2-1 & 1-2 also exist

Δ preserves grading

Homology is an invariant

$\chi(Kh(L)) = \hat{J}(L)$

graded Euler char. normalized Jones poly.

Complex split in 2 $\Rightarrow$ even & odd homology! — determined by grading

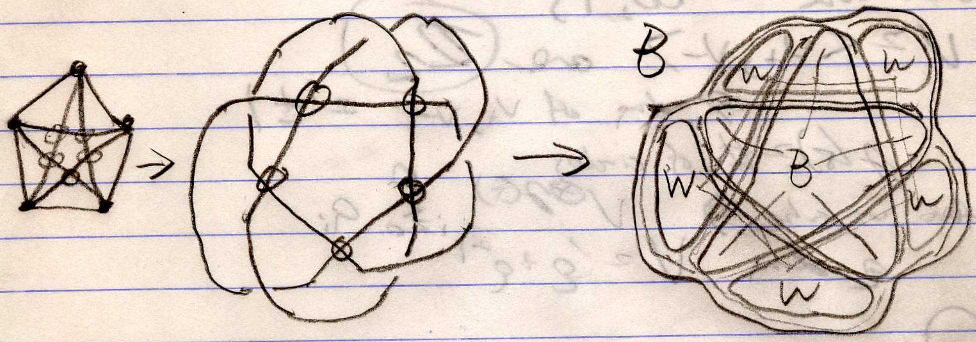
$4 \nmid \deg \hat{J}(L)$, $4 \mid \deg \hat{J}(L)$

$Kh_o = 0$ $Kh_e = 0$

Homology determined completely in $\mathbb{Z}_2$-complex by resolution cube & # of +/− crossings.

Examples

K_5 <u>is</u> checkerboard colorable



Knot on the torus (trefoil with 1 virtualized crossing) is <u>not</u>