TOM CAPILCE PG1
(TML)

Tutte and Dichromatic polynomials of graphs and their associated link Homfly polynomials

Before talk, write this:

Stuff: c(G) = connected components   n(G) = e(G) – v(G) + c(G)

Tutte
If G has one edge, t(G, x, y) = x if it's a bridge and t(G, x, y) = y if it's a loop.
If G has an edge e that is neither a bridge nor a loop,
T(G, x, y) = t(G – e, x, y) + t(G.e, x, y).
If G has two or more edges and e is a bridge, t(G, x, y) = xt(G.e, x, y).
If G has two or more edges and e is a loop, t(G, x, y) = yt(G – e, x, y).

T(G, x, y) = sum F in E(G) of (x -1)^(c(F) – c(E(G)) x (y – 1)^n(F)

T(G, x, y) = $\sum_{F \subseteq E(G)} (x-1)^{c(F)-c(E(G))} (y-1)^{n(F)}$

Homfly

If L is the trivial knot, P(L, x, y, z) = 1.
If L+, L-, Lo are like in the picture,
xP(L+, x, y, z) + yP(L-, x, y, z) + zP(Lo, x, y, z) = 0
If L and L' are isotopic, P(L, x, y, z) = P(L', x, y, z)
If L is split into L1 and L2, then P(L, x, y, z) = (-(x + y)/z)P(L1, x, y, z)P(L2, x, y, z)
If L is the connected sum of L1 and L2, P(L, x, y, z) = P(L1, x, y, z)P(L2, x, y, z)

P(L, x, y, z) =

Dichromatic polynomial for weighted graphs

For a weighted graph, let w be a function mapping E(G) into some commutative ring. Let m be the number of edges, and n(G) be the nullity of G. Let ~~po(G)~~ c(G) be the number of connected components of G. Let G: S be the subgraph of G with includes all the vertices of G but only the edges of S.

Q(G, t, z) = sum over S in E(G) product over S t^po(G: S) z^n(G: S)

Q(G, t, z) = $\sum_{S \subseteq E(G)} \prod_{S} t^{c(G:S)} z^{n(G:S)}$

If m = 0, Q(G: t, z) = t^n
If G is disconnected, Q is the product of the Qs of each component

If G has an edge e that is not a loop, $Q(G: t, z) = Q(G - e: t, z) + w(e)Q(G/e: t, z)$
If e is a loop, then $Q(G: t, z) = (1 + w(e)z)Q(G - e: t, z)$
If e is a bridge, then $Q(G: t, z) = (w(e) + t)Q(G.e: t, z)$

Note that if $w(e) = 0$, $Q(G; t, z) = Q(G - e; t, z)$.

Setup for theorem 1!

We're gonna take a graph and make a link diagram D(G) out of it in this way:

Construct the Medial Graph M(G) of G. Take the graph, draw circles around each vertex that connect, direct so that black face is on the left, and then do figure 1 to each of the vertices. Note that each vertex of M(G) corresponds to an edge of G.

PICTURE 1

Note that each finite face of G corresponds to an unknotted component of D(G) that is oriented clockwise. Recall that v(G) = number of vertices of G and e(G) = number of edges of G and c(G) = number of connected components of G.

Theorem 1.

Let G be a connected plane graph. For all nonzero numbers x, y, and z,

$$P(D(G), x, y, z) = (y/z)^{(v(G) - 1)} (-z/x)^{(e(G))}\, t(G, -x/y, 1 - ((xy + y^2)/z^2)$$

Proof.

We proceed by induction on the number of edges of G. Everything is in picture 3.

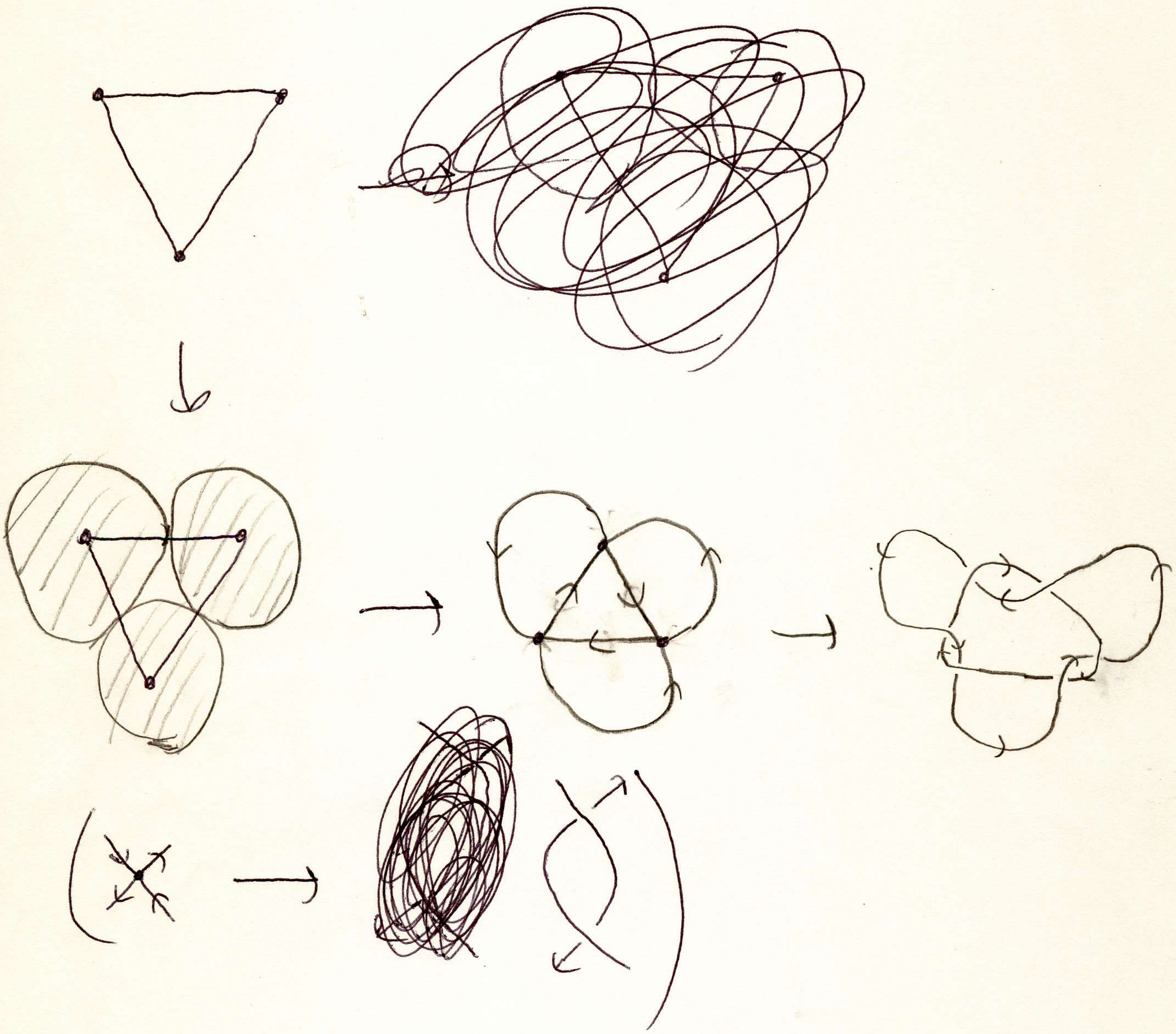
Base case: G has one edge e. See PICTURE 3 and 4

Setup for theorem 2

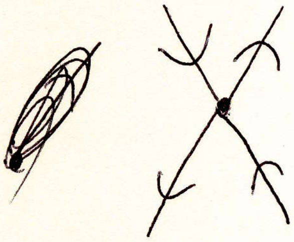
We can extend Theorem 1 to weighted graphs of four weights y/z, x/z, z/y, and z/x. Let a, b, c, and d be the numbers of edges with these weights respectively. Construct a medial graph like before. Now, in the medial graph, each vertex is associated with an edge. We construct D(G) as in PICTURE TWO and then we get this fun result:

$$(-x/z)^a(-y/z)^b(-x/y)^c(-y/x)^d P(D(G), x, y, z) = (-(x + y)/z)^{-1}Q(G, -(x + y)/z, -(x + y)/z)$$

1)

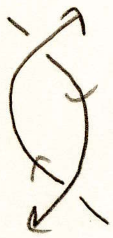


2

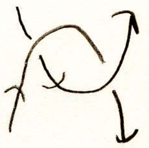
 — VERTEX OF EDGE e

$w(e) = \frac{y}{z}$

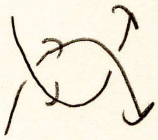
$w(e) = \frac{x}{z}$



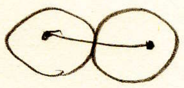
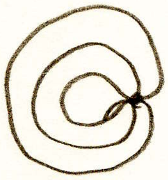
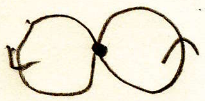
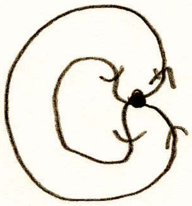
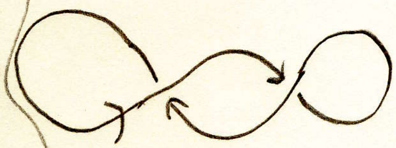
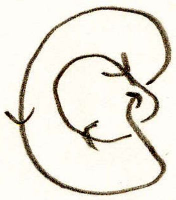
$w(e) = \frac{z}{y}$



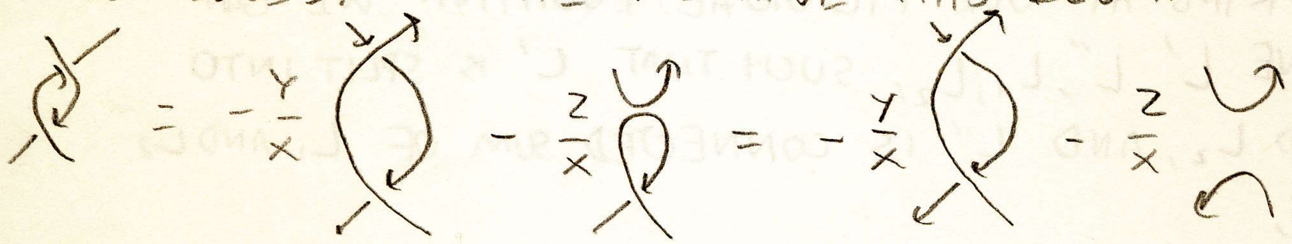
$w(e) = \frac{z}{x}$



3 BASE CASE

	BRIDGE	LOOP
G		
M(G)		
D(G)		
P(D(G))	1	$-\frac{z}{x} - \left(\frac{x+y}{z}\right)\left(-\frac{y}{x}\right) = -\frac{z}{x} + \frac{y}{z} + \frac{y^2}{zx}$
$T\left(G, -\frac{x}{y}, 1-\frac{xy+y^2}{z^2}\right)$	$-\frac{x}{y}$	$1-\frac{xy+y^2}{z^2}$
$\left(\frac{y}{z}\right)^{v(G)-1}$	$\frac{y}{z}$	1
$\left(-\frac{z}{x}\right)^{e(G)}$	$-\frac{z}{x}$	$-\frac{z}{x}$
PRODUCT OF ABOVE	1	$\frac{y}{z} + \frac{y^2}{xz} - \frac{z}{x}$

(4) LET e BE AN EDGE OF G. SUPPOSE G HAS 2 OR MORE EDGES. SO LIKE WE HAVE THIS EQUATION:



CASE 1: e IS NIETHER A LOOP NOR A BRIDGE.

PICTORIAL EQUATION IS NOW

$$P(D(G),x,y,z) = -\frac{y}{x}P(D(G.e),x,y,z) - \frac{z}{x}P(D(G-e),x,y,z)$$

AND $G.e$ AND $G-e$ ARE CONNECTED.

BY INDUCTIVE HYPOTHESIS,

$$P(D(G.e),x,y,z) = \left(\frac{y}{z}\right)^{v(G)-2}\left(-\frac{z}{x}\right)^{e(G)-1} t\left(G.e, -\frac{x}{y}, 1-\frac{xy+y^2}{z^2}\right)$$

$$P(D(G-e),x,y,z) = \left(\frac{y}{z}\right)^{v(G)-1}\left(-\frac{z}{x}\right)^{e(G)-1} t\left(G-e, -\frac{x}{y}, 1-\frac{xy+y^2}{z^2}\right)$$

$$\Rightarrow P(D(G),x,y,z) = \left(\frac{y}{z}\right)^{v(G)-1}\left(-\frac{z}{x}\right)^{e(G)}\left[t(G.e,\dots)+t(G-e,\dots)\right]$$

$$= \left(\frac{y}{z}\right)^{v(G)-1}\left(-\frac{z}{x}\right)^{e(G)} t(G,\dots).$$

CASE 2: e IS A BRIDGE. $D(G)$ AND $D(G.e)$ ARE ISOTOPIC.

SO $P(D(G),x,y,z) = P(D(G.e),x,y,z)$

SO BY INDUCTION

$$P(D(G),x,y,z) = \left(\frac{y}{z}\right)^{v(G)-2}\left(-\frac{z}{x}\right)^{e(G)-1} t\left(G.e, -\frac{x}{y}, 1-\frac{xy+y^2}{z^2}\right)$$

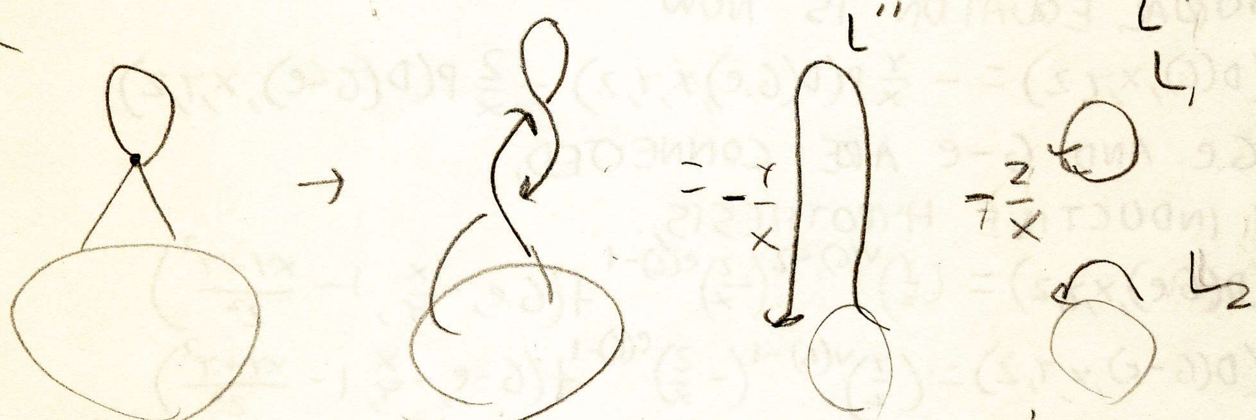
SO BY A PROPERTY OF TUTTE, WE'RE GOLDEN.

CASE 3: e IS A LOOP OF G.

LOOKING AT OUR PICTORIAL EQUATION, WE CAN DEFINE L', L'', L_1, L_2, SUCH THAT L' IS SPLIT INTO L_1 AND L_2, AND L'' IS CONNECTED SUM OF L_1 AND L_2.

THEN,

$$P(D(G),x,y,z) = \left(-\frac{y}{x}\right)P(L',x,y,z) - \left(\frac{z}{x}\right)P(L'',x,y,z)$$



L'' IS ISOTOPIC TO $D(G-e)$, AND

$$P(L',x,y,z) = -\frac{x+y}{z}P(L'',x,y,z)$$

SO

$$P(D(G),x,y,z) = \left(\frac{xy+y^2}{xz} - \frac{z}{x}\right)P(D(G-e),x,y,z)$$

SINCE $G-e$ IS CONNECTED, BY OUR INDUCTION HYPOTHESIS,

$$P(D(G-e),x,y,z) = \left(\frac{y}{z}\right)^{v(G)-1}\left(-\frac{z}{x}\right)^{e(G)-1} t(G-e,\dots)$$

AND $$t(G-e,\dots) = \frac{t(G,\dots)}{1-\frac{xy+y^2}{z^2}}$$

SO $$P(D(G),x,y,z) = \underbrace{\frac{\frac{xy+y^2}{xz} - \frac{z}{x}}{1-\frac{xy+y^2}{z^2}}}_{= -\frac{z}{x}} \left(\frac{y}{z}\right)^{v(G)-1}\left(-\frac{z}{x}\right)^{e(G)-1} t(G,\dots)$$