Strong Maps of Matroids: The Problems

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Our purpose was to find some relation between the Tutte polynomial of a strong map of matroids, (M, M'), from the bond matroid, M, of the dual of the original ribbon graph to the circuit matroid, M', of the ribbon graph

$$T(M, M'; \alpha, \beta, \gamma) = \sum_{F \subseteq E} \alpha^{r(M') - r_{M'}(F)} \beta^{n_M(F)} \gamma^{r(M) - r(M') - (r_M(F) - r_{M'}(F))} \tag{1}$$

and the Bollobás-Riordan polynomial of the ribbon graph

$$R(M'; X, Y, Z) = \sum_{F \subseteq E} X^{r(M') - r_{M'}(F)} Y^{n_{M'}(F)} Z^{k(F) - bc(F) + n_{M'}(F)}. \tag{2}$$

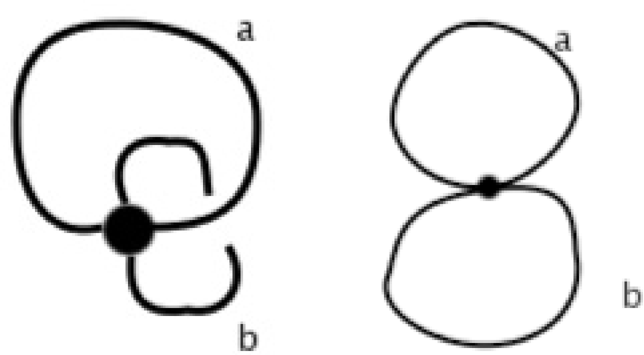

Figure 1: Our first ribbon graph and its dual.

As an example, if we have figure 1 for our first M' and M, respectively, then the polynomials are $R(M'; X, Y, Z) = 1 + 2Y + Y^2Z^2$ and $T(M, M'; \alpha, \beta, \gamma) = \gamma^2 + 2\gamma + 1$. So, we notice that $T(M, M'; \alpha, \beta, \gamma) = \gamma^2 R(M', \alpha, \gamma^{-1}, 1)$. Similarly for figure 2 we have $R(M'; X, Y, Z) = X + 2 + XY + 3Y + Y^2Z^2$ and $T(M, M'; \alpha, \beta, \gamma) = \alpha\gamma^2 + 2\gamma^2 + \alpha\gamma + 3\gamma + 1$, and so we again get $T = \gamma^2 R(\alpha, \gamma^{-1}, 1)$.

Note that in both cases we have α in the X-slot. This is because in both polynomials the powers of these two variables are $r(M') - r_{M'}(F)$. This suggests that our relation should be of the form $T = \beta^{b_0}\gamma^{c_0} R(\alpha, \beta^{b_2}\gamma^{c_2}, \beta^{b_3}\gamma^{c_3})$ for some integers b_i, c_i. In fact, for most of

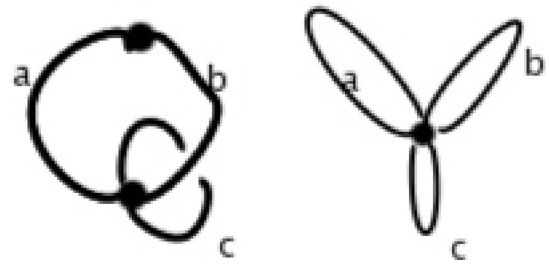

Figure 2: Our second ribbon graph and its dual.

the examples we die where anything of this form worked, we were getting T equalling either $\gamma^{r(M)-r(M')}R(\alpha,\gamma^{-1},1)$ or $\gamma^{r(M)-r(M')}R(\alpha,\beta,1)$.

Dr. Chmutov suggested to us that we look at Theorem 2 in Bollobás and Riordan's paper "A polynomial of graphs on surfaces" [BR3]. To explain this theorem we'll need some preliminaries. And we'll be making some simplifications since this paper is not solely interested in oriented ribbon graphs, as we are.

- Let $\mathcal{G}$ denote the set of isomorphism classes of connected and oriented ribbon graphs.
- For the coefficient of Y^iZ^j in the Bollobás-Riordan polynomial, R, we'll write R_{ij}. So each R_{ij} is a map from $\mathcal{G}$ to $\mathbb{Z}[X]$.
- Given a ring $\mathbf{R}$ and $x \in \mathbf{R}$, $R_{ij}(x)$ will mean the map from $\mathcal{G}$ to $\mathbf{R}$ given by composing the map R_{ij} with the natural ring homomorphism from $\mathbb{Z}[X]$ to $\mathbf{R}$ which maps X to x.

Also, note that this theorem is for the polynomial defined by

$$R(\mathbf{G}) = \sum_{F\subseteq E}(X-1)^{r(M')-r_{M'}(F)}Y^{n_{M'}(F)}Z^{k(F)-bc(F)+n_{M'}(F)}$$

rather than (2), although I suspect that this shouldn't affect us much.

Theorem 1 *Let $\mathbf{R}$ be a commutative ring, x an element of $\mathbf{R}$, and ϕ a map from $\mathcal{G}$ to $\mathbf{R}$ satisfying*

$$\phi(\mathbf{G}) = \begin{cases} \phi(\mathbf{G}/e)+\phi(\mathbf{G}-e) & \textit{if } e \textit{ is ordinary,} \\ x\phi(\mathbf{G}/e) & \textit{if } e \textit{ is a bridge.} \end{cases}$$

Then there are elements λ_{ij} for $i \geq 0$, $0 \leq j \leq i$, such that

$$\phi = \sum_{i,j}\lambda_{ij}R_{ij}(x).$$

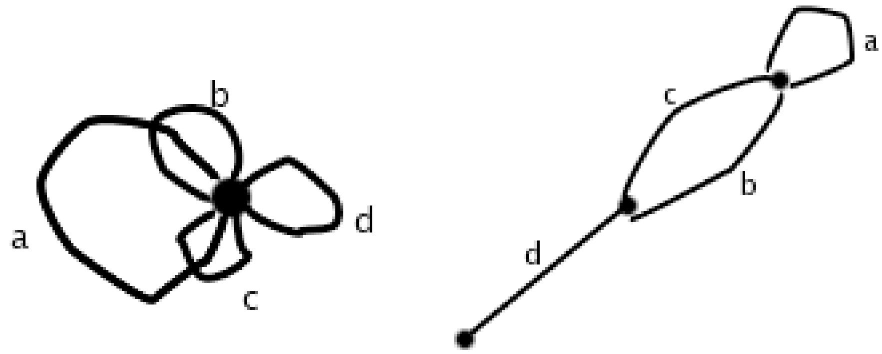

Figure 3: A problematic example

Now in our specific example, $\mathbf{R}$ should be $\mathbb{Z}[\alpha, \beta, \gamma]$ and $x = \alpha$. But note that in the first example above $2Y$ gets sent to 2γ, and in the next example, from figure 3, $4Y$ gets sent to $3\gamma + \beta\gamma^2$.

The general problem we seemed to have had is exemplified here in figure 3, for which many of the calculations can be seen in table 1. Note that, although with just one loop in M' we always get the monomial Y, in the Tutte polynomial we get γ for the loops a, b, or c, and $\beta\gamma^2$ for the loop d.

	$\emptyset$	a	b	c	d	...	bcd	acd	abd	abc	E
$r_{M'}$	0	0	0	0	0		0	0	0	0	0
$n_{M'}$	0	1	1	1	1		3	3	3	3	4
k	1	1	1	1	1		1	1	1	1	1
bc	1	2	2	2	2		2	2	2	4	3
	1	Y	Y	Y	Y		Y^3Z^2	Y^3Z^2	Y^3Z^2	Y^3	Y^4Z^2
r_M	0	1	1	1	0		2	2	2	1	2
n_M	0	0	0	0	1		1	1	1	2	2
	γ^2	γ	γ	γ	$\beta\gamma^2$		β	β	β	$\beta^2\gamma$	β^2

Table 1: The polynomial calculations for figure 3