Knots and Graphs

VIGRE Working Group [Summer 2006]

MATH 693, call number 11615-4, 3 credits

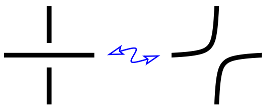
Instructor: ***Sergei Chmutov***

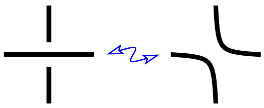
RESEARCH PROJECTS

Introduction.

The Kauffman bracket and the Jones polynomial [Ka1]

Let L be a link diagram.

A-splitting: 

B-splitting: 

A *state* S is a choice of either A- or B-splitting at every classical crossing.

$\alpha(S) = \#(\text{of } A\text{-splittings in } S)$

$\beta(S) = \#(\text{of } B\text{-splittings in } S)$

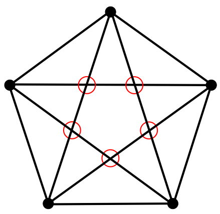
$\delta(S) = \#(\text{of circles in } S)$

$$[L](A,B,d) \;:=\; \sum_S A^{\alpha(S)}\, B^{\beta(S)}\, d^{\delta(S)-1}$$

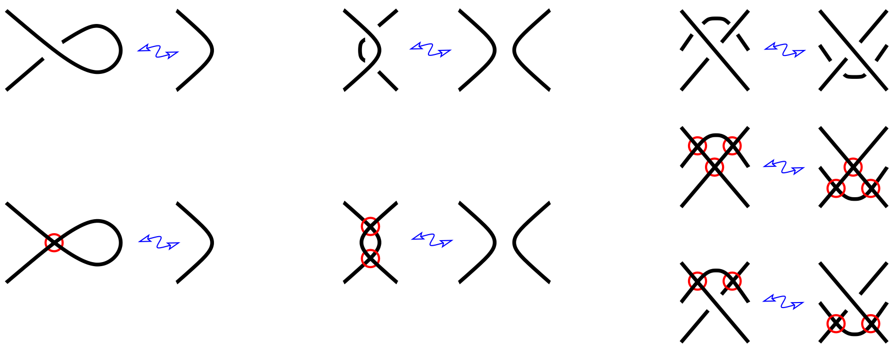
$$J_L(t) \;:= (-1)^{w(L)} t^{3w(L)/4} [L](t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2})$$

Virtual links [Ka3, Kam]

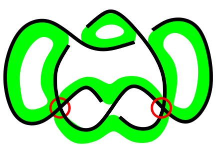
Virtual crossings

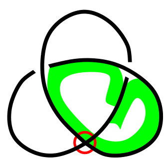


Reidemeister moves

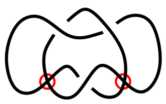
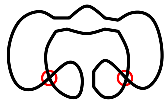
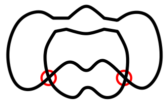
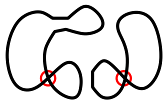
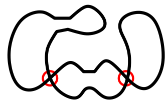
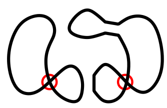
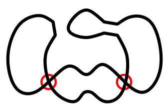
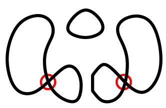
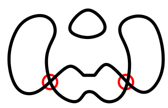


Checkerboard colorable virtual links (Naoko Kamada)

colorable 

 non-colorable

Example

				
(α,β,δ)	$(3,0,1)$	$(2,1,2)$	$(2,1,2)$	$(1,2,1)$
				
	$(2,1,2)$	$(1,2,1)$	$(1,2,3)$	$(0,3,2)$

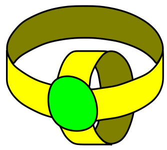
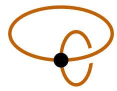
$$[L] = A^3 + 3A^2Bd + 2AB^2 + AB^2d^2 + B^3d; \qquad J_L(t) = 1$$

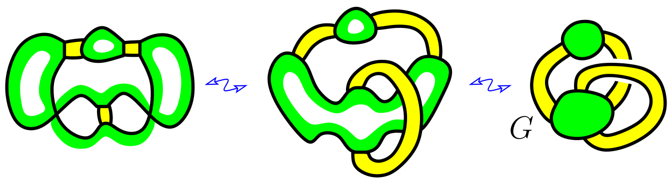
Ribbon graphs (B. Bollobás and O. Riordan [BR2, BR3])

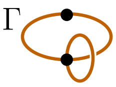
A ribbon graph G is a surface represented as union of vertices-discs (green) and edges-ribbons (yellow) such that

- discs and ribbons intersect by disjoint line segments,
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- every edge contains exactly two such line segments.

Examples

$G =$  $\Gamma =$ 



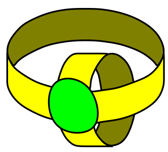
Γ 

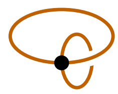
The Bollobás-Riordan polynomial [BR2, BR3]

Let

- G be a ribbon graph;
- $v(G)$ be the number of its vertices;
- $e(G)$ be the number of its edges;
- $k(G)$ be the number of components of G;
- $r(G) := v(G) - k(G)$ be the *rank* of G;
- $n(G) := e(G) - r(G)$ be the *nullity* of G;
- $\mathrm{bc}(G)$ be the number of connected components of the boundary of G;

Example

$G =$ 

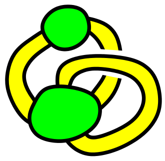
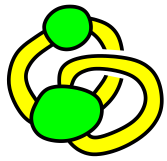
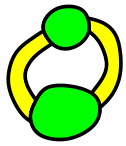
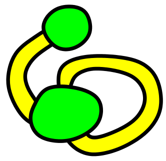
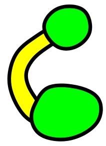
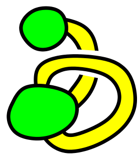
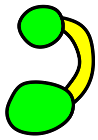
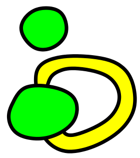
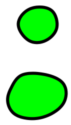
$\Gamma =$ 

$$\begin{aligned} v(G) &= 1 \\ e(G) &= 2 \\ k(G) &= 1 \\ r(G) &= 0 \\ n(G) &= 2 \\ \mathrm{bc}(G) &= 1 \end{aligned}$$

$$R_G(x,y,z) = 1 + 2y + y^2z^2$$

$$\boxed{R_G(x,y,z) \ := \sum_F x^{r(G)-r(F)} y^{n(F)} z^{k(F)-\mathrm{bc}(F)+n(F)}}$$

Example

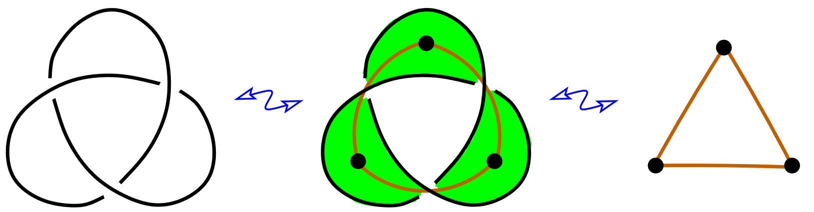
				
(k,r,n,bc)	$(1,1,2,1)$	$(1,1,1,2)$	$(1,1,1,2)$	$(1,1,0,1)$
				
	$(1,1,1,2)$	$(1,1,0,1)$	$(2,0,1,3)$	$(2,0,0,2)$

$$R_G(x,y,z) \ = \ y^2z^2 + 3y + 2 + xy + x$$

Relations to the Tutte polynomial

- $R_G(x-1, y-1, 1) = T_\Gamma(x,y)$.
- If G is planar (genus zero): $R_G(x-1, y-1, z) = T_\Gamma(x,y)$.

Thistlethwaite's Theorem [Th, Ka1, Ja] *Up to a sign and multiplication by a power of t the Jones polynomial $J_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_\Gamma(-t, -t^{-1})$.*



The theorem was generalized to non-alternating links (using signed graphs in [Ka2] and using the Bollobás-Riordan polynomial for ribbon graphs in [DFKLS]; and checkerboard colorable virtual links in [CP].

Theorem [CP]. *Let L be an alternating virtual link and G_L be the corresponding ribbon graph. Then*

$$[L](A,B,d) \ = A^{r(G)} B^{n(G)} d^{k(G)-1} \, R_{G_L}\left(\frac{Bd}{A}, \frac{Ad}{B}, \frac{1}{d}\right)$$

Project 1. ***Virtual links.*** *erica j. Whitaker, Stephen Swihart, Jeremy Voltz.*

Generalize the theorem of [CP] to arbitrary virtual links. Probably one should use the non-orientable signed version of the Bollobás-Riordan polynomial from [BR3] and generalize the construction of a ribbon graph corresponding to a link diagram from [DFKLS]. Are there any other virtual link invariants which can be obtained from the Bollobás-Riordan polynomial?

Project 2. ***Parallel connection of ribbon graphs.*** *Nishali Mehta, Nicholas Kefauver, Alex Mominee.*

Tensor product $G_1 \otimes G_2$ of two (ribbon) graphs G_1 and G_2 can be defined as a result of replacement of each edge of G_1 by a copy of the graph G_2. There is a formula [Hug, JVW, Wo] for the Tutte polynomial of $G_1 \otimes G_2$ in terms of the Tutte polynomials of G_1 and G_2. This project aims to generalize this formula to the the Bollobás-Riordan polynomial of ribbon graphs. Some special cases of such formulas for ribbon graphs were found in [Mof, p.8-9].

Project 3. ***Strong maps of matroids.*** *Charles Estill, Daniel Grollmus, Min Ro.*

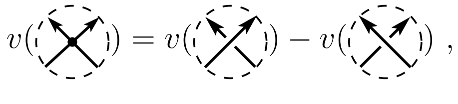
For each ribbon graph, regarded as a graph embedded into a surface, there is a dual ribbon graph, embedded into the same surface. The matroids of these two graphs form the so called *strong map* of matroids. For a strong map of matroids there is a generalization of the Tutte polynomials which is a polynomial in three variables [LV1, LV2]. Investigate the relations of this polynomial with the Bollobás-Riordan polynomial of the initial ribbon graph. For general intoroduction to matroids see [Ox, Wh].

Project 4. ***Hamilton cycles.*** *James Sharpnack, Justin Wiser.*

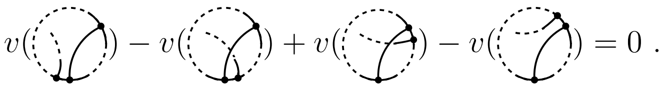
The number of Hamilton cycles in a graph can be found using tensors from linear algebra [Zo]. Namely, with each vertex of a graph we associate a tensor of the valency equal to the valency of the vertex. If two vertices are connected by an edge we contract the corresponding tensor factors. After all contractions we will get a number which is equal to the number of Hamilton cycles of the graph. Study this construction in the special case of the Cayley graph of a finite group. Determine the tensors in terms of the group. Is it possible to simplify the corresponding tensors in this case?

Project 5. ***Polyak-Viro formulas for Vassiliev invariants.*** *Michael (Cap) Khoury, Alfred Rossi.*

The notion of *finite type* (*Vassiliev*) knot invariants appeared around 1990 in V. Vassiliev's investigations of discriminants in the (infinite-dimensional) spaces of smooth maps from one manifold into another. The idea is to extend a knot invariant v to singular knots with double points according to the rule



known as *Vassiliev's skein relation.* An invariant v is called a Vassiliev invariant of order $\leqslant n$ if its extension vanishes on all singular knots with more than n double points. The main combinatorial object underlying this theory is a function on chord diagrams (a circle with a bunch of chords in it) satisfying the following 4-term relation. Such a function is called a *weight system.*



For a reader-friendly introduction to this theory see the draft of our book [CD]. A way to describe Vassiliev invariants of a knot in terms of its Gauss diagram was found by M. Polyak and O. Viro (see [CD, Ch.13]). Find the Polyak-Viro formulas for the the coefficients of the Jones polynomial, the Conway polynomial, other polynomial knot invariants.

Project 6. ***Khovanov homology of links.*** *Shaun Van Ault, Brad Waller.*

Several years ago M. Khovanov [Kho] discovered a homology theory for links whose graded Euler characteristic is equal to the Jones polynomial. More precisely, for every link L he defined a sequence of vector spaces (abelian groups) $\mathcal{H}^{i,j}(L)$ such that

$$J_L(q) = \sum_{i,j} (-1)^i q^j \dim(\mathcal{H}^{i,j}(L)) \ .$$

There are two excellent expositions of this theory [BN, Vi]. This project is a continuation of the last summer project. Namely last summer we found two simplicial complexes whose homology are the top and the bottom lines of the Khovanov homology. Do the same for other lines of the Khovanov complex. This will require a generalization of the results of Bae and Morton [BM] to the other coefficients of the Jones polynomial.

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