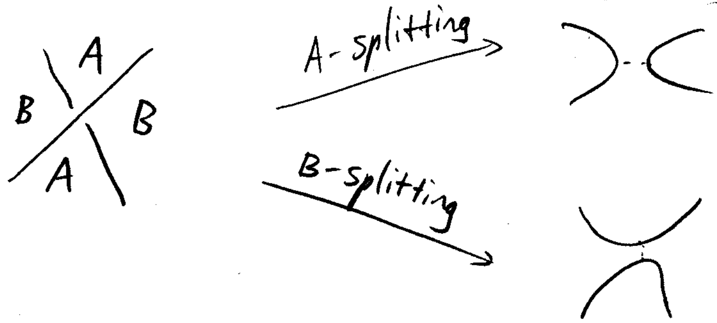
Coefficients of the Kauffman Bracket.

Let L be a link diagram with n crossings, with a chosen order on the crossings.



Let $S = \{ (X_1, X_2, \ldots, X_n) : X_i = A \text{ or } B \}$, the set of states.

For $s \in S$, let $\alpha(s) = \#$ of A's,

$\beta(s) = \#$ of B's,

At the i^{th} crossing, perform an A-splitting (resp. B-splitting) if $s_i = A$ (resp. B). Let

$\delta(s) = \#$ of circles in diagram after splitting according to s.

Univariate Kauffman Bracket:

$$\langle L \rangle \in \mathbb{Z}[A, A^{-1}],$$

$$\langle L \rangle (A) = \sum_{s \in S} A^{\alpha(s) - \beta(s)} (-A^2 - A^{-2})^{\delta(s) - 1}$$

For any state, s, $\alpha(s) + \beta(s) = n$, so the s-term can be written $f(s) := A^{n - 2\beta(s)} (-A^2 - A^{-2})^{\delta(s)-1}$

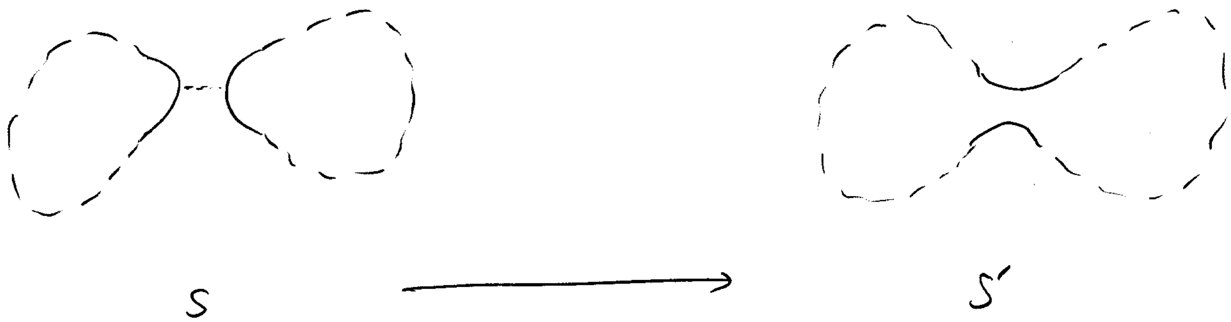
$$= \sum_{i=0}^{\delta(s)-1} (-1)^{\delta(s)-1} \binom{\delta(s)-1}{i} A^{n-2\beta(s)} A^{2(\delta(s)-1-i)} A^{-2i}$$

$$= \sum_{i=0}^{\delta(s)-1} (-1)^{\delta(s)-1} \binom{\delta(s)-1}{i} A^{n-2\beta(s)+2\delta(s)-2-4i}$$

Consider a state s', obtained from s by changing one A-splitting to a B-splitting. There are two cases:

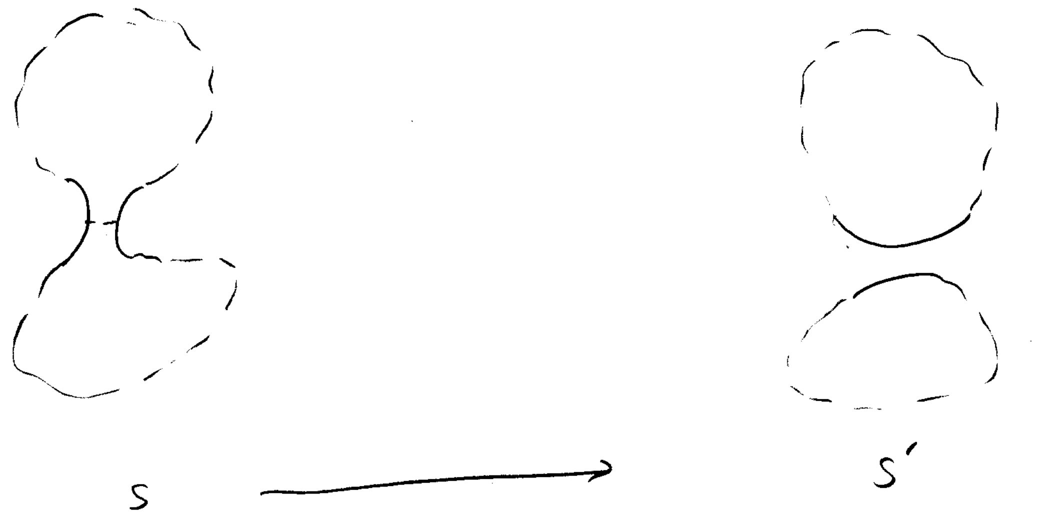
<u>Def</u>: An A-splitting is outer if it joins two different circles. It is inner if it joins the same circle.

Outer :



$$\beta(s') = \beta(s) + 1$$
$$\delta(s') = \delta(s) - 1$$

Inner :



$$\beta(s') = \beta(s) + 1$$
$$\delta(s') = \delta(s) + 1$$

<u>Def</u> : $\operatorname{maxdeg}(s) := \max\deg(f(s)) = n - 2\beta(s) + 2\delta(s) - 2$

Note, in $f(s)$, the degree of terms decreases by 4.

Now, if $s \longrightarrow s'$ is obtained through an outer splitting,

$$\operatorname{maxdeg}(s') = n - 2\beta(s') + 2\delta(s') - 2 = n - 2\beta(s) - 2 + 2\delta(s) - 2 - 2$$
$$= \operatorname{maxdeg}(s) - 4$$

If $s \longrightarrow s'$ is obtained through an inner splitting,

$$\max\deg(s') = n - 2\beta(s') + 2\delta(s') - 2 = n - 2\beta(s) - 2 + 2\delta(s) + 2 - 2$$
$$= \max\deg(s).$$

Hence, $\max\deg(s)$ is non-increasing as A-splittings change to B-splittings.

<u>Def</u>: $s_A = (A, A, \ldots, A)$, total A-splitting.

then, from the above, $\max\deg(s_A) \geq \max\deg(\langle L \rangle)$

$$\max\deg(s_A) = n + 2\delta(s_A) - 2$$

Let $d = \delta(s_A)$.

Write $\langle L \rangle (A) = \sum_{i=0}^{M} c_i(L) A^{\deg_i(L)}$,

where M is chosen large enough.

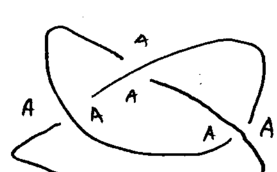
So, $\deg_0(L) = n + 2d - 2$, and $\deg_i(L) = n + 2d - 2 - 4i$

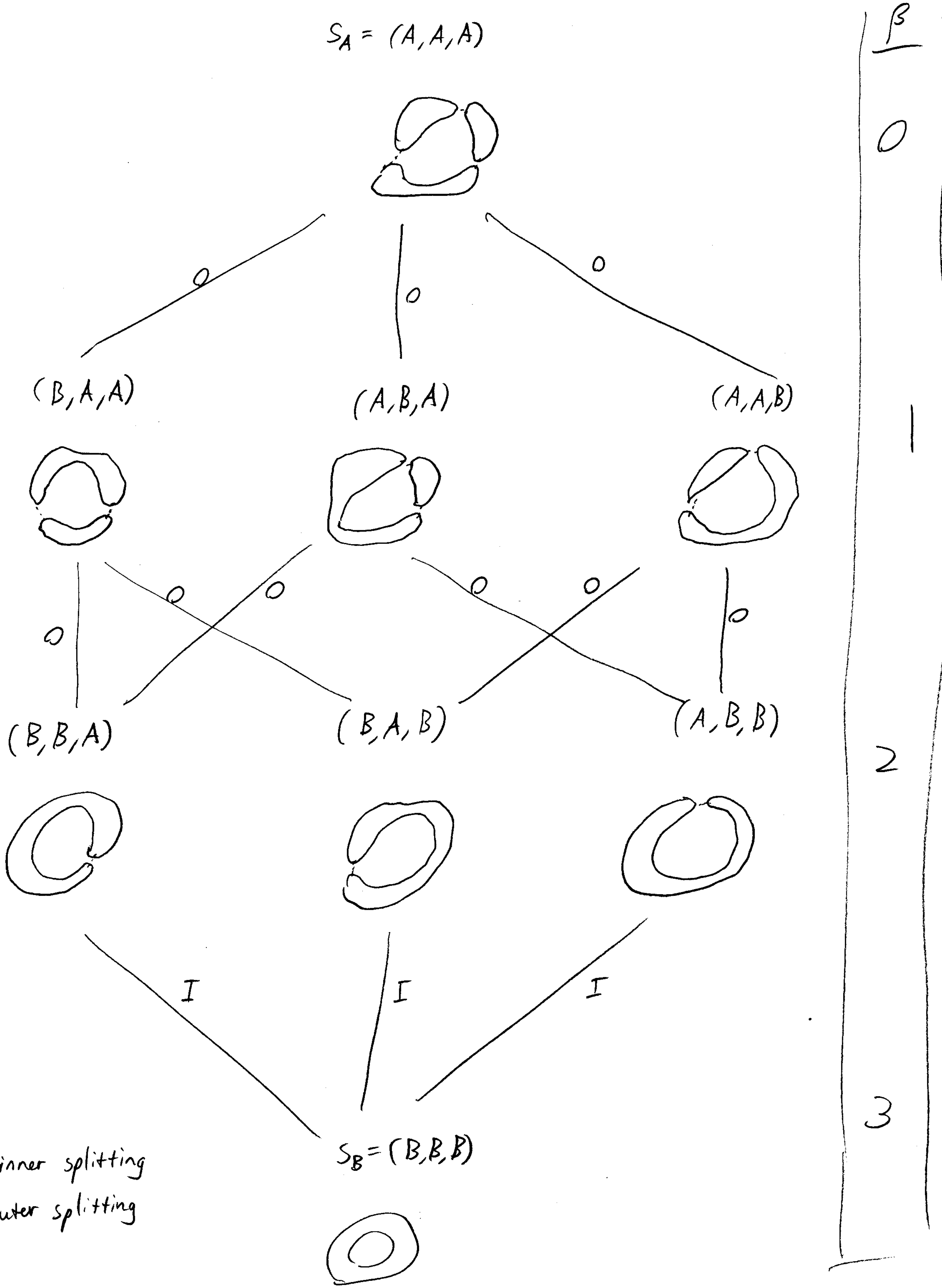
We allow any $c_0(L)$ to be 0, including $c_0(L)$.

Resolution Graph of L:

The states $s \in S$ may be represented as the vertices of an n-cube, displayed so that height corresponds to $\beta(s)$. An edge is drawn from s to s' iff $s \longrightarrow s'$ is obtained by changing one A-splitting to a B-splitting. Label each splitting as inner/outer.

<u>Ex</u>: Trefoil:





I - inner splitting
O - outer splitting

Lemma: In any path from s_A to s, the number of outer splittings remains constant.

Proof: Consider $\delta(s)$. This number is independent of path (just count state circles), but from the previous, along with a straightforward induction,

$$\delta(s) = \delta(s_A) + (\#\text{ inner splittings}) - (\#\text{ outer splittings}),$$

where we choose any resolution path.

Let $\eta = \#$ outer splittings in the path.

Then $\#$ inner splittings $= \beta(s) - \eta$, so

$$\delta(s) = \delta(s_A) + \beta(s) - 2\eta,$$

showing that η is independent of path.

Let $\eta(s) = \#$ of outer splittings in any resolution path starting at s_A.

Theorem: A state s contributes exactly

$$(-1)^{d+\beta(s)-1}\binom{d+\beta(s)-1-2\eta(s)}{j-\eta(s)}$$

to $c_j(L)$.

In particular, if $\eta(s) > j$, s makes no contribution.

Cor: $$c_0(L) = \sum_{\substack{s \in S \\ \eta(s)=0}} (-1)^{d+\beta(s)-1}$$ [Bae, Morton]

Proof of Theorem:

For a state s, with $\eta(s) = k$, we have

$$\max\deg(s) = \max\deg(s_A) - 4k,$$

by induction on the resolution path of s.

Now, s_A contributes the i^{th} coefficient of $f(s_A)$ to c_j, and that exponent is: $n + 2d - 2 - 4j$.

Consider $f(s) = \sum_{i=0}^{\delta(s)-1} (-1)^{\delta(s)-1} \binom{\delta(s)-1}{i} A^{n-2\beta(s)+2\delta(s)-2-4i}$

$$= \sum_{i=0}^{d+\beta-2\eta-1} (-1)^{d+\beta-2\eta-1} \binom{d+\beta-2\eta-1}{i} A^{n-2\beta+2(d+\beta-2\eta)-2-4i}$$

Find the term with exponent $n+2d-2-4j$:

$$\cancel{n}+\cancel{2d}-\cancel{2}-4j = \cancel{n}-\cancel{2\beta}+\cancel{2d}+\cancel{2\beta}-4\eta-\cancel{2}-4i$$

$$4i = 4j - 4\eta$$

$$i = j - \eta$$

$\Rightarrow$ contribution of s to c_j is:

$$(-1)^{d+\beta(s)-2\eta(s)-1} \binom{d+\beta(s)-1-2\eta(s)}{j-\eta(s)}$$

$$= (-1)^{d+\beta(s)-1} \binom{d+\beta(s)-1-2\eta(s)}{j-\eta(s)}$$

So, we have the following formula:

$$\boxed{c_j(L) = \sum_{\substack{s\in S \\ \eta(s)\le j}} (-1)^{d+\beta(s)-1} \binom{d+\beta(s)-1-2\eta(s)}{j-\eta(s)}}$$

Recursive Formulae for $c_j(L)$:

I. If $L = L_1 \sqcup L_2$, then

$$c_j(L) = \sum_{i=0}^{j} c_i(L_1)\, c_{j-i}(L_2)$$

(follows directly from $\langle L \rangle = \langle L_1 \rangle \langle L_2 \rangle$

II. If c is a crossing, let L_c^A be the diagram obtained from L by splitting c as an A-splitting, and let L_c^B be the diagram obtained by splitting c as a B-splitting.

Then $$c_j(L) = \begin{cases} c_j(L_c^A) + c_j(L_c^B), & c \text{ inner} \\ c_j(L_c^A) + c_{j-1}(L_c^B), & c \text{ outer} \end{cases}$$

This follows from $\langle L \rangle = A \langle L_c^A \rangle + A^{-1} \langle L_c^B \rangle$

if $\langle L \rangle$ has n crossings and d state circles in s_A,

then $\langle L_A^c \rangle$ has $n-1$ crossings and d state circles,

so $\max\deg(s_A) = n-1+2d-2$ in L_A^c.

Multiply by A, and the coefficients line up.

$\langle L_B^c \rangle$ has $n-1$ crossing. If c is inner (relative to s_A), then there are $d+1$ state circles,

So $\max\deg(s_A) = n-1+2d+2-2 = n+2d-1$

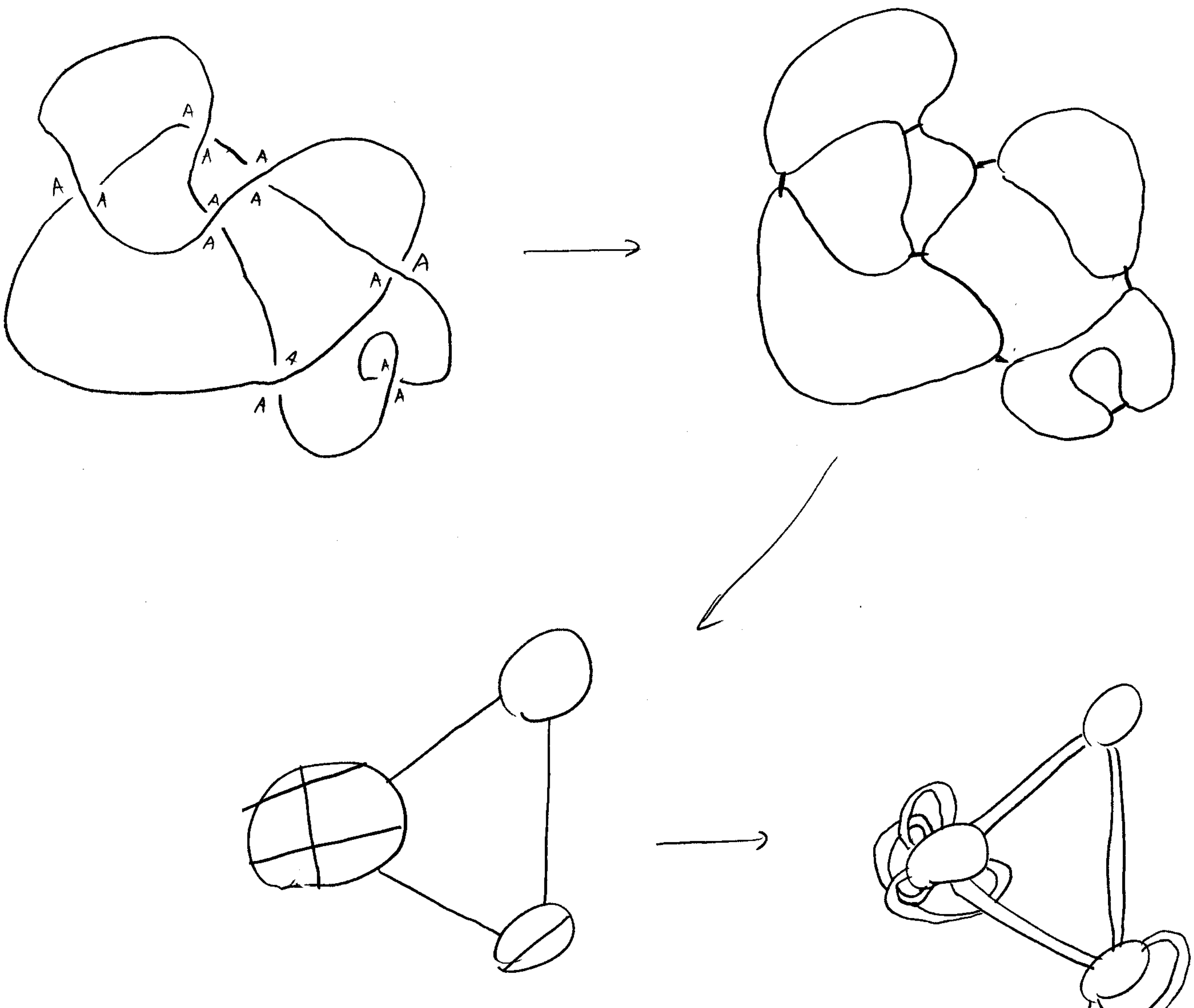
Multiply by A^{-1}, and the coefficients line up.

If c is outer, there are $d-1$ state circles so $\max\deg(s_A) = n-1 + 2d-2-2 = n+2d-5$.

Multiplying by A^{-1} yields $n+2d-6$, the next highest degree. Thus, all degrees are shifted by 1 (i.e., 1 unit of 4).

Graphical version, and future direction:

For a link, L, draw s_A as a ribbon graph, with edges corresponding to A-splittings:



Γ_L

Would like to find a nice formula based of Γ_L for each c_j. But, computing $R(\Gamma_L)$ is out of the question, since this exponential in the number of edges, and we might as well have computed $\langle L \rangle$ instead.

B-M describes how to obtain $c_0(L)$ in a nice way:

Let Γ_L' be Γ_L with only inner edges. Write

$$\Gamma_L = \Gamma_1 \sqcup \Gamma_2 \sqcup \ldots \sqcup \Gamma_k \qquad \text{(one for each vertex).}$$

Then $$c_0(L) = (-1)^{d-1} f(\Gamma_1) \cdot \ldots \cdot f(\Gamma_k),$$

where $f(\Gamma_i) = \sum_C (-1)^{|C|}$, the sum taken over sets of edges C of Γ_i that are independent, i.e., no pair of edges has alternating endpoints.

This method can be further refined by creating an appropriate cell complex for which $f(\Gamma_i)$ is just the Euler characteristic. It is conjectured that these cell complexes all have the homotopy type of a wedge of spheres of the same dimension, so that their reduced homology is concentrated in one dimension. If true, then the dimension is an invariant of L.

Progress in finding analogous results for $c_1, c_2, \ldots$ has not been forthcoming.