What is a virtual link?

Virtual links differ from ordinary knot diagrams by presence of virtual crossings, which are not crossings, but defects of our two-dimensional picture.

virtual crossings are encircled for emphasis.

<u>ordinary knot diagrams</u>

unknot

trefoil

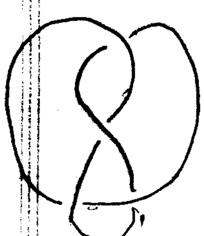
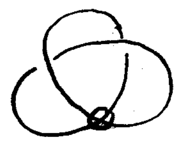
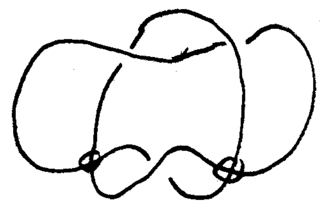


figure eight

<u>virtual link diagrams</u>

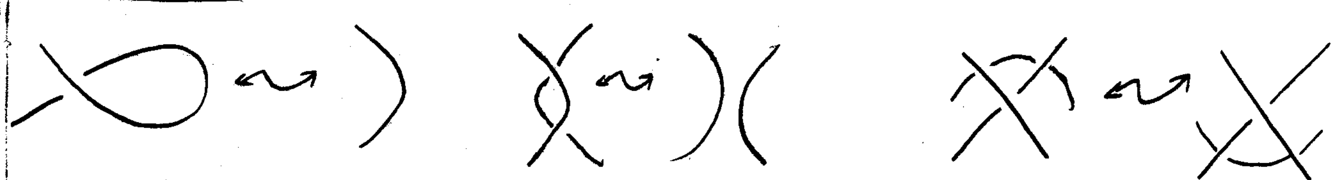


← not alternating

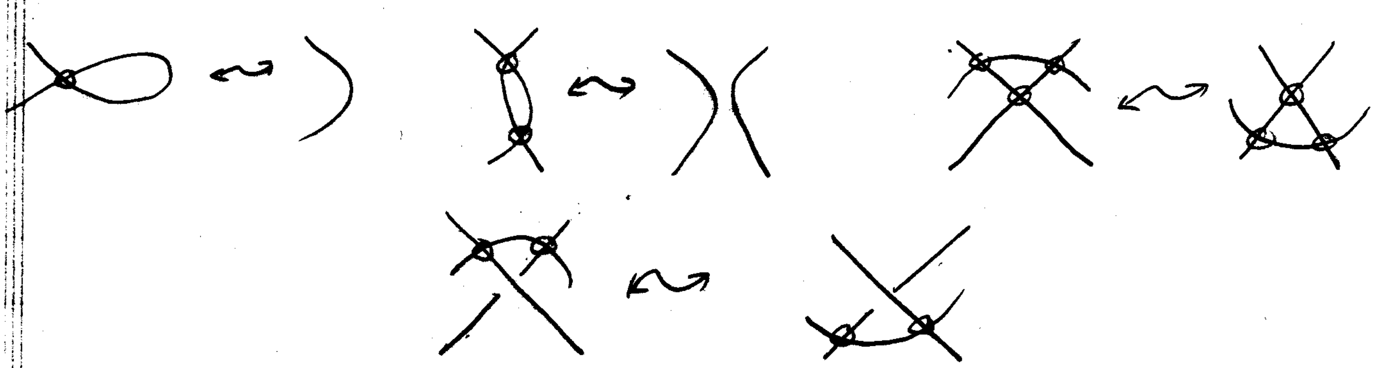


← <u>alternating</u>

Virtual link diagrams are considered modulo the <u>classical</u> Reidemeister moves:



& the <u>virtual</u> Reidemeister moves:



The Kaufman Bracket

Consider two ways of resolving a classical crossing:

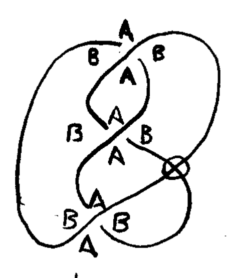
A splitting: 

B splitting: 

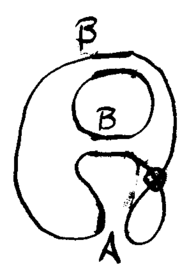
Given a virtual link diagram L, A **state** S of L is a choice of splitting for each classical crossing of L.

Denote by $S(L)$ the set of the states of L.
If L has n classical crossings, then $|S(L)| = 2^n$.

Let $\alpha(S)$ = # of A-splittings in a state S,
$\beta(S)$ = # of B-splittings in a state S,
& $\delta(S)$ = # of components by splitting according to S.

Ex: 

L



S

$\alpha(S) = 1$
$\beta(S) = 2$
$\delta(S) = 2$

Def: The Kaufmann bracket of a diagram is a polynomial in three variables A, B, d defined by the formula:

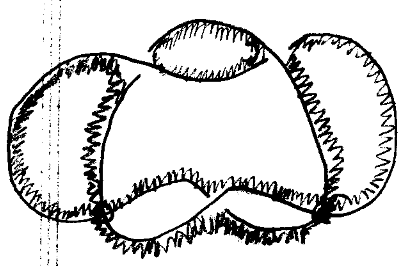
$$[L](A,B,d) = \sum_{s \in S(L)} A^{\alpha(S)} B^{\beta(S)} d^{\delta(S)-1}$$

Ribbon Graphs

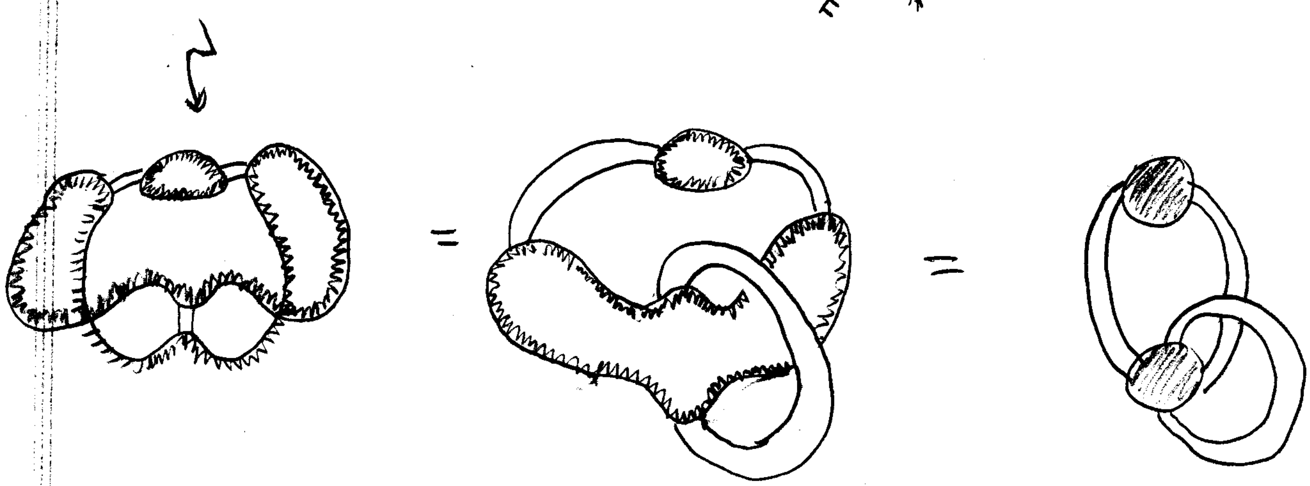
(Informally, a ribbon graph is a set of vertices as rigid 'discs' and edges as 'ribbons'. We only care about it as a 2-dimensional figure with a boundary.)

Customary construction for alternating virtual links

Fact: An alternating virtual link is checkerboard colorable. (Kauffman)



to construct the ribbons:

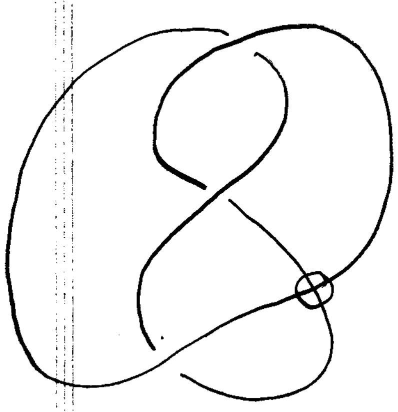


Just as Thistlethwaite's theorem relates an alternating classical link to an associated planar graph, Chmutov and Pak found a relation between any alternating virtual link and its associated ribbon graph.

What about non-checkerboard-colorable virtual links?

(We can't naturally construct the ribbon graph as before because without checkerboard colorability, vertices (discs) are not well-defined.)

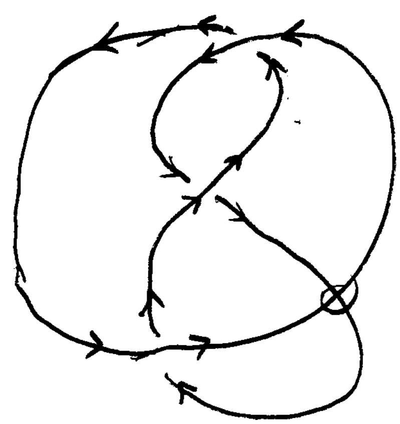
~~But there is a natural construction that we discovered~~: OUR CONSTRUCTION:



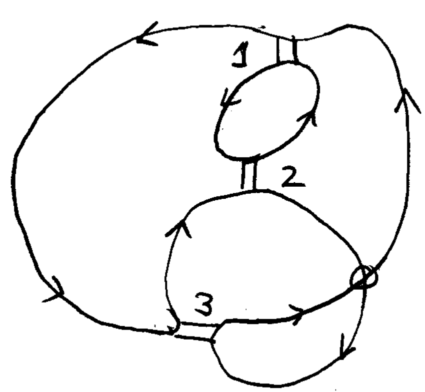
this virtual link is not checkerboard colorable.

Construction:

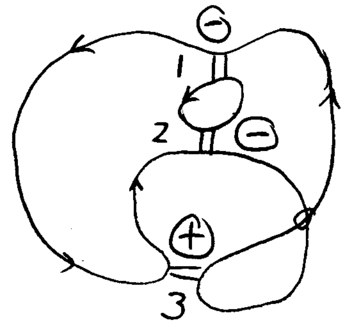
1. Orient the link (direction doesn't matter)



2. Split to preserve orientation, add ribbons, and number.

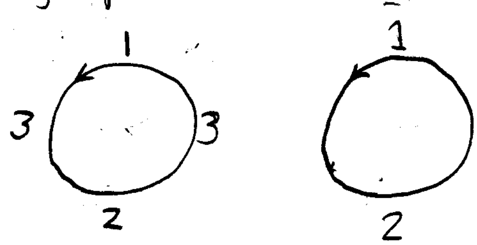


3. Label each ribbon either + or - if it was an A split or B split, respectively.

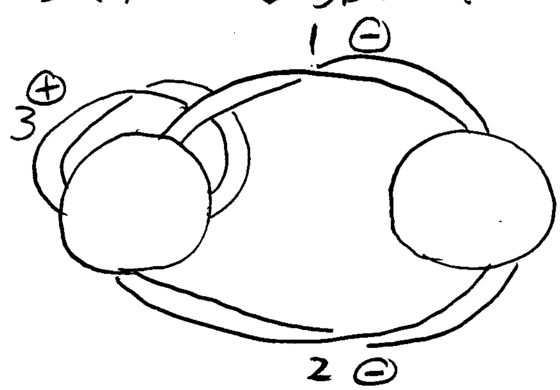


Without the ribbons this is a state of the diagram. Call this the main state, $\bar{S}$. In this case, $\bar{S}$ = BBA.

4. Create a disc for each component of this state and orient each of them counterclockwise. Number around the vertices to preserve the cyclic order around the components of $\bar{S}$. (starting point doesn't matter)



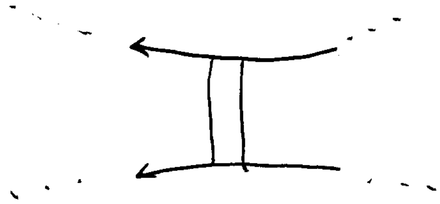
5. Add ribbons, noting sign, and TWIST EVERY RIBBON!



Thus we get a signed ribbon graph, $\hat{G}$.

Why do we twist all of the Ribbons?

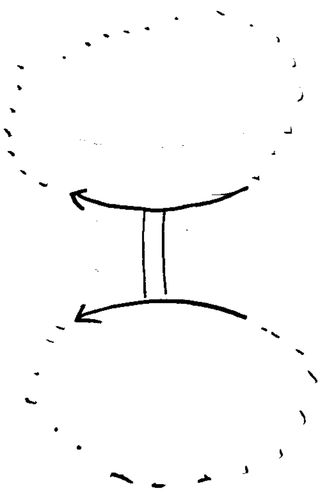
Since we preserve our orientation when we split all of the crossings, with our ribbon we always get something that looks like this:



(the arrows will always be going the same direction)

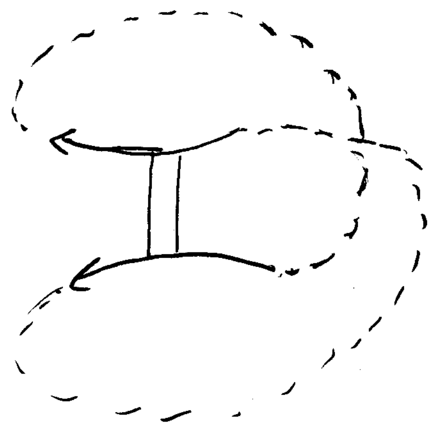
There are two cases for such a split:

I



or

II



In our construction, we oriented every disc the same direction, counterclockwise.

In case I, it is obvious that the ribbon must be twisted to satisfy this requirement.

In case II, by noticing the order in which we pass across the edges of the ribbon, we see the twist:

Technical Stuff

Just as Chmutov and Pak found a relation between an alternating virtual link and its associated ribbon graph, we have discovered a relation between **any** virtual link diagram and its signed ribbon graph just described.

Theorem: Let L be a virtual link diagram and $\hat{G}_L$ be the corresponding signed ribbon graph. Then

$$[L](A,B,d) = A^{e(G)-r(G)} B^{r(G)} d^{k(G)-1} R_{\hat{G}_L}\left(\frac{Ad}{B}, \frac{Bd}{A}, \frac{1}{d}\right)$$

where the signed Bollobás-Riordan polynomial is given by

$$R_{\hat{G}_L}(x,y,z) = \sum_{F \in \mathcal{F}(\hat{G}_L)} x^{r(G)-r(F)+s(F)} y^{n(F)-s(F)} z^{k(F)-bc(F)+n(F)},$$

and where:

$\mathcal{F}(G)$ is the set of spanning subgraphs of G,

$s(F) = \dfrac{e_-(F) - e_-(\bar{F})}{2}$,

$e_-(F) =$ # negative edges in F,

$\bar{F}$ is the spanning subgraph composed of all edges not in F,

$v(G) =$ # vertices of G,

$e(G) =$ # edges of G,

$k(G) =$ # components of G,

$r(G) = v(G) - k(G)$

$n(G) = e(G) - r(G)$

$bc(G) =$ # connected boundary components of G.

Sketch of the Proof

Let L be a virtual link and $\hat{G}_L$ be the associated signed ribbon graph. Let $\bar{S}$ be the main state of $\hat{G}_L$.

There is a natural one-to-one correspondence between the states $S \in S(L)$ and spanning subgraphs $F \in \mathcal{F}(\hat{G}_L)$. Namely, if a crossing in S is split differently than it is in $\bar{S}$, include it in the spanning subgraph F. If a crossing is split the same way for both S & $\bar{S}$, don't include it in F. This gives the associated subgraph F of S.

Given that $F \in \mathcal{F}(\hat{G}_L)$ is associated to $S \in S(L)$ as given above, consider the term

$$x^{r(G)-r(F)+s(F)}\, y^{n(F)-s(F)}\, z^{k(F)-bc(F)+n(F)} \text{ of } R_{\hat{G}}(x,y,z)$$

Substitute in $x = \frac{Ad}{B}$, $y = \frac{Bd}{A}$, $z = \frac{1}{d}$ and multiply by the term $A^{e(G)-r(G)} B^{r(G)} d^{k(G)-1}$ as in the theorem to get (after much cancellation).

$$A^{e(G)-r(F)-n(F)+2s(F)}\, B^{r(F)+n(F)-2s(F)}\, d^{bc(F)-1}.$$

We want to show that this is equal to the Kauffman bracket term $A^{\alpha(S)} B^{\beta(S)} d^{\delta(S)-1}$, where S is the state associated to F.

$bc(F) = \delta(S)$ by the construction of our ribbon graph.

And proving that $r(F)+n(F)-2s(F) = \beta(S)$ is a simple counting argument.