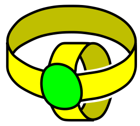
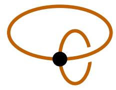
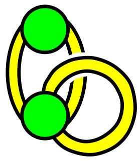
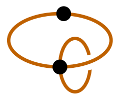
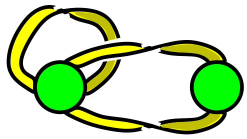
Ribbon graphs (B. Bollobás and O. Riordan [BR2, BR3])

A ribbon graph G is a surface represented as union of vertices-discs and edges-ribbons such that

- discs and ribbons intersect by disjoint line segments,
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- every edge contains exactly two such line segments.

$G =$  $\Gamma =$ 

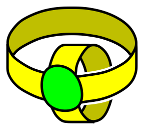
$G =$  $\Gamma =$ 

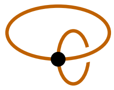
$G =$ 

The Bollobás-Riordan polynomial [BR2, BR3]

- $v(G)$ be the number of vertices of G;
- $e(G)$ be the number of its edges of G;
- $k(G)$ be the number of components of G;
- $r(G) := v(G) - k(G)$ be the *rank* of G;
- $n(G) := e(G) - r(G)$ be the *nullity* of G;
- $\mathrm{bc}(G)$ be the number of connected components of the boundary of G;

$$R_G(x,y,z) := \sum_F x^{r(G)-r(F)} y^{n(F)} z^{k(F)-\mathrm{bc}(F)+n(F)}$$

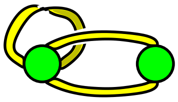
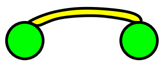
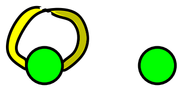
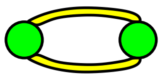
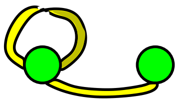
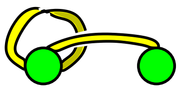
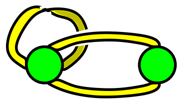
$G =$ 

$\Gamma =$ 

$$v(G) = 1,\quad e(G) = 2,\quad k(G) = 1,\quad r(G) = 0,\quad n(G) = 2,\quad \mathrm{bc}(G) = 1$$

$$R_G(x,y,z) = 1 + 2y + y^2z^2$$

Example

				
(k,r,n,bc)	$(2,0,0,2)$	$(1,1,0,1)$	$(1,1,0,1)$	$(2,0,1,2)$
				
	$(1,1,1,2)$	$(1,1,1,1)$	$(1,1,1,1)$	$(1,1,2,1)$

$$R_G(x,y,z) = x + 2 + xyz + y + 2yz + y^2z^2\ .$$

Relations to the Tutte polynomial

$R_G(x-1, y-1, 1) = T_\Gamma(x,y)$. If G is planar (genus zero): $R_G(x-1,y-1,z) = T_\Gamma(x,y)$.

References

[BR2] B. Bollobás, O. Riordan, *A polynomial of graphs on orientable surfaces*, Proc. London Math. Soc. **83** (2001) 513–531.

[BR3] B. Bollobás, O. Riordan, *A polynomial of graphs on surfaces*, Math. Ann. **323**(1) (2002) 81–96.