Knots and Graphs

Working Group [Summer 2007]

MATH 693, call number 12009-3, 3 credits

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RESEARCH PROJECTS

Project 1. ***Virtual links.*** (Alfred Rossi, Michael (Cap) Khoury)

This continues the work of M. Khoury, A. Rossi, and J. Voltz from the last summer. The aim is to find new invariants of virtual links based on [ChVo, KR]. In [KR], M. Khoury and A. Rossi found two expressions for each coefficient of the Conway polynomial of a knot. For a classical knot, these expressions are equal, but we expect that similar expressions can be defined for virtual knots and in this case they would be different. This would lead to new invariants of virtual knots which vanish on classical knots. We would like to to investigate this situation in detail and obtain new virtual invariants if possible. References [Ka3, CP] contain further information about virtual links.

Project 2. ***Parallel connection of ribbon graphs.*** (Min Ro, Sam Fotis, Raeyong Kim)

This project was considered last summer but still remains unsolved.

The *tensor product* $G_1 \otimes G_2$ of two (ribbon) graphs G_1 and G_2 is defined as the result of replacing each edge of G_1 by a copy of the graph G_2. There is a known formula for the Tutte polynomial of $G_1 \otimes G_2$ in terms of the Tutte polynomials of G_1 and G_2, as described in [Hug, JVW, Wo]. This project aims to generalize that formula to the the Bollobás-Riordan polynomial of ribbon graphs. Some special cases of such formulas for ribbon graphs were found in [Mof, p.8-9].

New references [Br, DHH] concerning this problem may provide some hints toward a solution. This project is also related to project 6 concerning the Khovanov homology of links. See [Wa].

Project 3. ***Strong maps of matroids.*** (Jonathan Belcher, Ryan Greene)

This is also a continuation of a project from last summer.

For each ribbon graph, regarded as a graph embedded in a surface, there is a dual ribbon graph, embedded in the same surface. The matroids of these two graphs form a so-called *strong map* of matroids. For such a strong map, the Tutte polynomials has been generalized to a polynomial in three variables [LV1, LV2]. We propose a systematic investigation of the relations of this generalized Tutte polynomial with the Bollobás-Riordan polynomial of the initial ribbon graph. For an introduction to matroids see [Ox, Wh].

In a related direction, we plan consider a newly discovered generalized duality of ribbon graphs with respect to a subset of edges. Is there any strong map of the corresponding matroids? How are the generalized Tutte polynomial and the Bollobás-Riordan polynomial related in this case?

Project 4. ***Reconstruction of ribbon graphs*** (Gunnu Parinandi, Douglas Schaefer, Dong Du, Matthew Sequin)

This problem was posted by Professor N. Robertson as an interesting open question.

The *Reconstruction Conjecture* in graph theory says that graphs are determined uniquely by the list of their proper subgraphs. We plan to investigate this famous conjecture in the case of ribbon graphs. Even those ribbon graphs with one vertex provide an interesting starting point that is not yet resolved. The relevant references are [Ke, McK, NW].

Project 5. ***Lie algebra generalization of the Bollobás-Riordan polynomial.*** (Deepak Bal, Michael Chmutov, Angelo Nasca, Trent Ohl)

In [BR2, BR3], one motivation for introducing the Bollobás-Riordan polynomial of a ribbon graph was to find new *weight systems* for finite type knot invariants. The Bollobás-Riordan polynomial arose from consideration of the weight system for the Lie algebra $\mathfrak{gl}(N)$. However there are weight systems coming from other semisimple Lie algebras. The aim of this project is to generalize the Bollobás-Riordan polynomial by using other semisimple Lie algebras. As a first step we propose to do this for the Lie algebra $\mathfrak{so}(N)$.

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