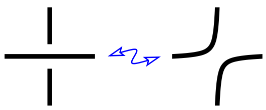
Jones polynomial

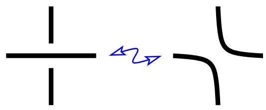
bigskip ***The Kauffman bracket and the Jones polynomial*** [Ka1].

Let L be a link diagram.

A-splitting:



B-splitting:



A *state* S is a choice of either A- or B-splitting at every classical crossing.

$\alpha(S) = \#(\text{of } A\text{-splittings in } S)$

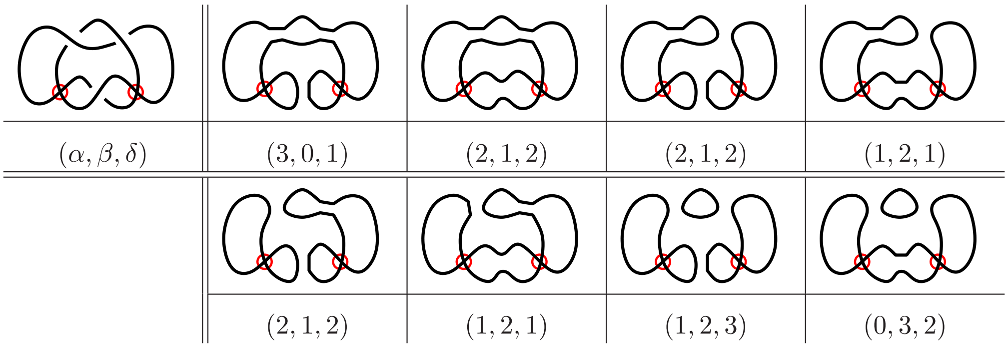
$\beta(S) = \#(\text{of } B\text{-splittings in } S)$

$\delta(S) = \#(\text{of circles in } S)$

$$[L](A,B,d) \;:=\; \sum_S A^{\alpha(S)}\, B^{\beta(S)}\, d^{\delta(S)-1}$$

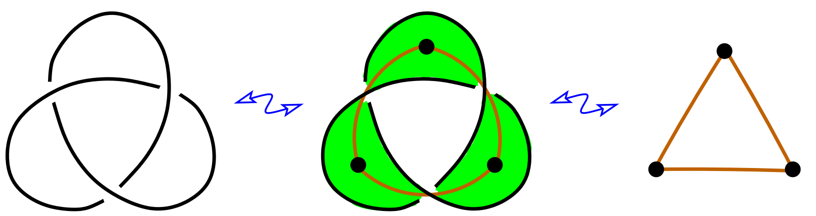
$$J_L(t) \;:= (-1)^{w(L)} t^{3w(L)/4} [L](t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2})$$

Example



$$[L] = A^3 + 3A^2Bd + 2AB^2 + AB^2d^2 + B^3d; \qquad J_L(t) = 1$$

Thistlethwaite's Theorem [Ka1] *Up to a sign and multiplication by a power of t the Jones polynomial $J_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_\Gamma(-t, -t^{-1})$.*



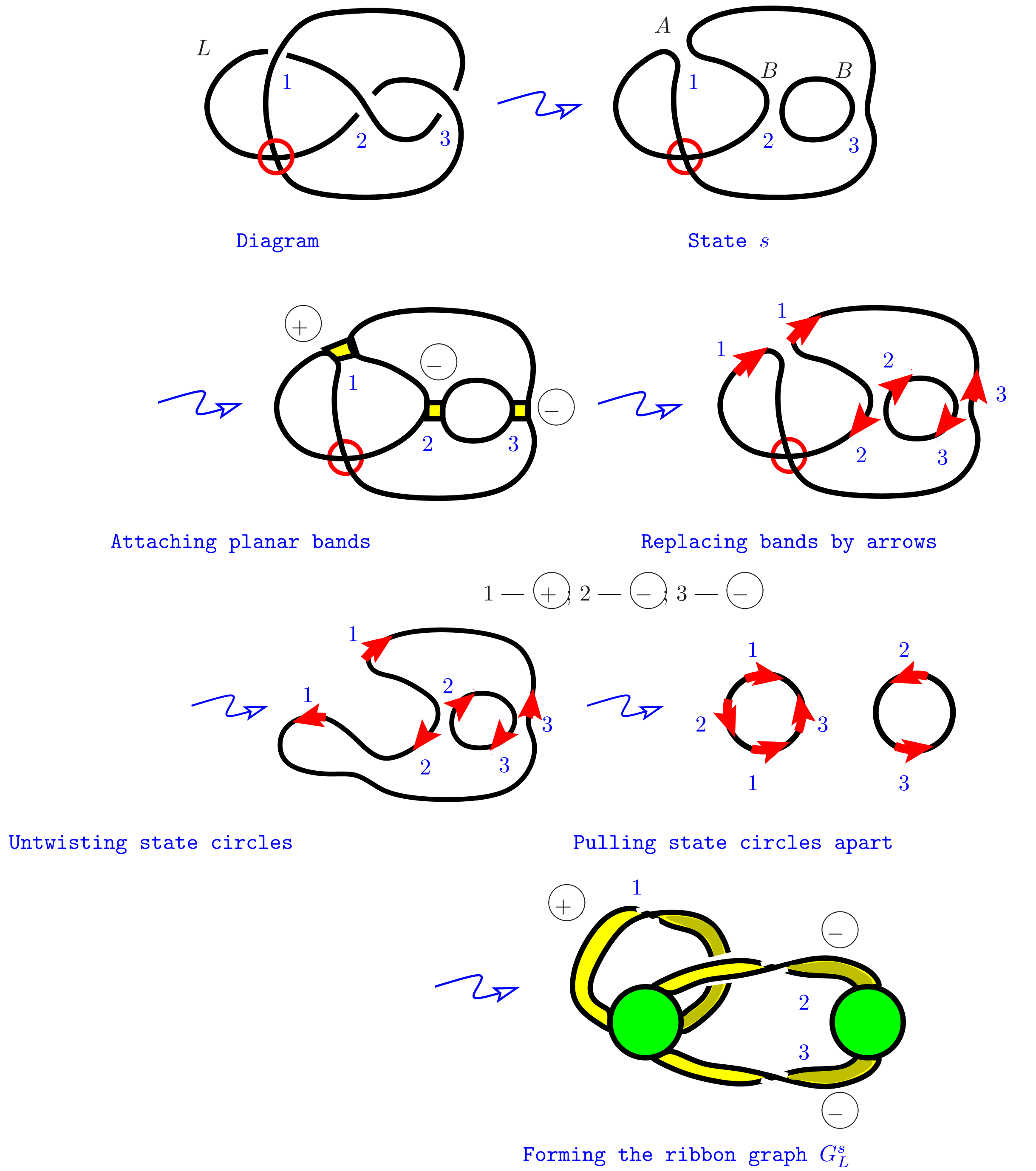
The theorem was generalized to non-alternating links using signed graphs in [Ka2] and using the Bollobás-Riordan polynomial for ribbon graphs in [DFKLS]; and to virtual links in [ChVo, Ch].

Theorem [Ch].

Let L be a virtual link diagram with e classical crossings, G_L^s be the signed ribbon graph corresponding to a state s, and $v := v(G_L^s)$, $k := k(G_L^s)$. Then $e = e(G_L^s)$ and

$$[L](A,B,d) = A^e \left(x^k y^v z^{v+1} R_{G_L^s}(x,y,z) \Big|_{x=\frac{Ad}{B},\ y=\frac{Bd}{A},\ z=\frac{1}{d}} \right).$$

Construction of a ribbon graph from a virtual link diagram



References

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[ChVo] S. Chmutov, J. Voltz, *Thistlethwaite's theorem for virtual links.* Journal of Knot Theory and Its Ramifications, **17**(10) (2008) 1189-1198; preprint `arXiv:math.GT/0704.1310`.

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[Ka1] L. H. Kauffman, *New invariants in knot theory*, Amer. Math. Monthly **95** (1988) 195–242.

[Ka2] L. H. Kauffman, *A Tutte polynomial for signed graphs*, Discrete Appl. Math. **25** (1989) 105–127.