The Las Vergnas polynomial

Reference: M. Las Vergnas [LV].

Matroid perspectives.
A bijection $M \to M'$ is called *matroid perspective* if any circuit of M is mapped to a union of circuites of M'. Equivalently,

$$r_M(X) - r_M(Y) \geqslant r_{M'}(X) - r_{M'}(Y) \qquad \text{for all} \quad Y \subseteq X.$$

Example.
For graphs G and G^* dually embedded in a surface, then the map of the bond matroid of G^* onto the circuit matroid of G, $\mathcal{B}(G^*) \to \mathcal{C}(G)$, is a matroid perspective.

Definition.

$$T_{M\to M'}(x,y,z) \ := \sum_{X\subseteq M} (x-1)^{r(M')-r_{M'}(X)}(y-1)^{n_M(X)} z^{(r(M)-r_M(X))-(r(M')-r_{M'}(X))}$$

Properties.

$$T_M(x,y) = T_{M\to M}(x,y,z) \ ;$$
$$T_M(x,y) = T_{M\to M'}(x,y,x-1) \ ;$$
$$T_{M'}(x,y) = (y-1)^{r(M)-r(M')} T_{M\to M'}(x,y,\tfrac{1}{y-1}) \ ;$$

For a ribbon graph G, let $t_G(x,y,z) := T_{\mathcal{B}(G^*)\to\mathcal{C}(G)}(x,y,z)$.

$$t_G = t_{G-e} + t_{G/e} \quad \textit{if } e \textit{ is neither a bridge nor a loop} \ ;$$
$$t_G = xT_{G/e} \quad \textit{if } e \textit{ is a bridge} \ ;$$
$$t_\bullet = 1 \ .$$

The relative Tutte polynomial

Reference: Y. Diao, G. Hetyei [DH].

Definition.

Let H be a subset of edges, 0-edges, of a graph G. For another subset, $F \subset G \setminus H$ let H_F be a graph obtained from G by deleting the edges in $\bar{F} := G \setminus (F \cup H)$ and contracting the edges from F.

$$T_H(G) \ := \sum_{F\subseteq G\setminus H} \Big(\prod_{e\in F} x_e\Big)\Big(\prod_{e\in \bar{F}} y_e\Big) X^{r(G)-r(F\cup H)} Y^{n(F)} \psi(H_F)$$

where ψ is a block-invariant function on graphs.

References

[DH] Y. Diao, G. Hetyei, *Relative Tutte polynomials for colored graphs and virtual knot theory*, Combinatorics, Probability and Computing **19** (2010) 343-369.

[LV] M. Las Vergnas, *On the Tutte polynomial of a morphism of matroids*, Annals of Discrete Mathematics **8** (1980) 7–20.