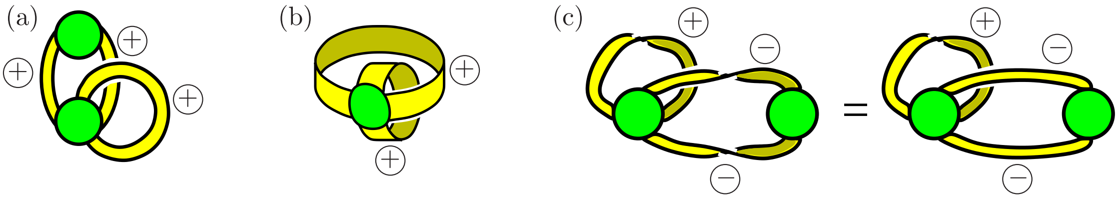
Ribbon graphs and the Bollobás-Riordan polynomial

Definition. A *ribbon graph* G is a surface (possibly non-orientable) with boundary, represented as the union of two sets of closed topological discs called *vertices* $V(G)$ and *edges* $E(G)$, satisfying the following conditions:

- these vertices and edges intersect by disjoint line segments;
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- every edge contains exactly two such line segments.

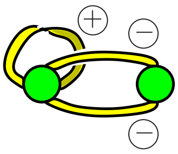
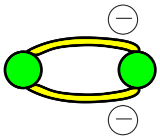
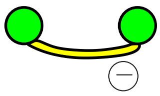
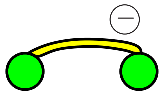
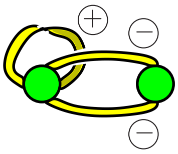
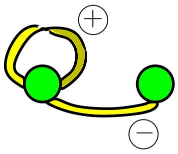
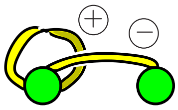
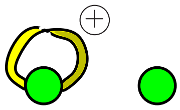


The Bollobás-Riordan polynomial

$$R_G(\{x_e, y_e\}, X, Y, Z) \;:=\; \sum_{F\subseteq G} \Big(\prod_{e\in F} x_e\Big)\Big(\prod_{e\in \overline{F}} y_e\Big) X^{r(G)-r(F)} Y^{n(F)} Z^{k(F)-\mathrm{bc}(F)+n(F)}$$

For signed graphs, we set
$$\begin{cases} x_+ = 1, & x_- = (X/Y)^{1/2}, \\ y_+ = 1, & y_- = (Y/X)^{1/2}. \end{cases}$$

Example.

				
(k,r,n,bc) term of R_G	(1,1,1,2) X	(1,1,0,1) 1	(1,1,0,1) 1	(2,0,0,2) Y
				
	(1,1,2,1) XYZ^2	(1,1,1,1) YZ	(1,1,1,1) YZ	(2,0,1,2) Y^2Z

$$R_G(X, Y, Z) = X + 2 + Y + XYZ^2 + 2YZ + Y^2Z$$

Properties.

$R_G = x_e R_{G/e} + y_e R_{G-e}$ if e is ordinary, that is neither a bridge nor a loop,

$R_G = (x_e + X y_e) R_{G/e}$ if e is a bridge.

$R_{G_1 \sqcup G_2} = R_{G_1 \cdot G_2} = R_{G_1} \cdot R_{G_2}$