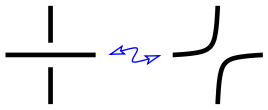
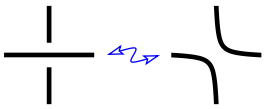
Jones polynomial

The Kauffman bracket and the Jones polynomial [Ka1].

Let L be a link diagram.

A-splitting: 

B-splitting: 

A *state* S is a choice of either A- or B-splitting at every classical crossing.

$\alpha(S) = \#(\text{of } A\text{-splittings in } S)$

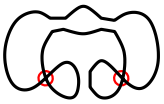
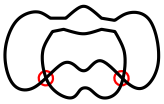
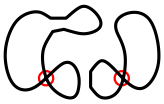
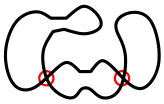
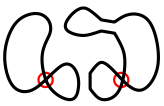
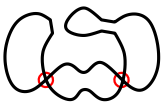
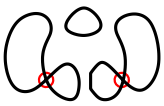
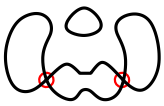
$\beta(S) = \#(\text{of } B\text{-splittings in } S)$

$\delta(S) = \#(\text{of circles in } S)$

$$[L](A,B,d) := \sum_S A^{\alpha(S)}\, B^{\beta(S)}\, d^{\delta(S)-1}$$

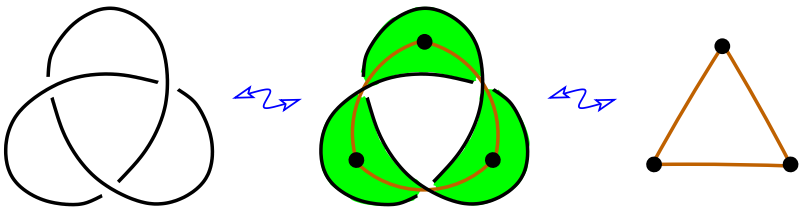
$$J_L(t) := (-1)^{w(L)} t^{3w(L)/4} [L](t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2})$$

Example

				
(α,β,δ)	$(3,0,1)$	$(2,1,2)$	$(2,1,2)$	$(1,2,1)$
				
	$(2,1,2)$	$(1,2,1)$	$(1,2,3)$	$(0,3,2)$

$$[L] = A^3 + 3A^2Bd + 2AB^2 + AB^2d^2 + B^3d; \qquad J_L(t) = 1$$

Thistlethwaite's Theorem [Ka1] *Up to a sign and multiplication by a power of t the Jones polynomial $J_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_\Gamma(-t,-t^{-1})$.*



The theorem was generalized to non-alternating links using signed graphs in [Ka2] and using the Bollobás-Riordan polynomial for ribbon graphs in [DFKLS]; and to virtual links in [ChVo, Ch].

Theorem [Ch].

Let L be a virtual link diagram with e classical crossings, G_L^s be the signed ribbon graph corresponding to a state s, and $v := v(G_L^s)$, $k := k(G_L^s)$. Then $e = e(G_L^s)$ and

$$[L](A,B,d) = A^e \left(x^k y^v z^{v+1} R_{G_L^s}(x,y,z) \Big|_{x=\frac{Ad}{B},\ y=\frac{Bd}{A},\ z=\frac{1}{d}} \right) .$$

Construction of a ribbon graph from a virtual link diagram

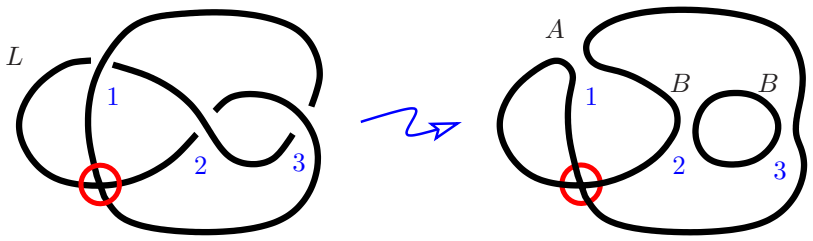
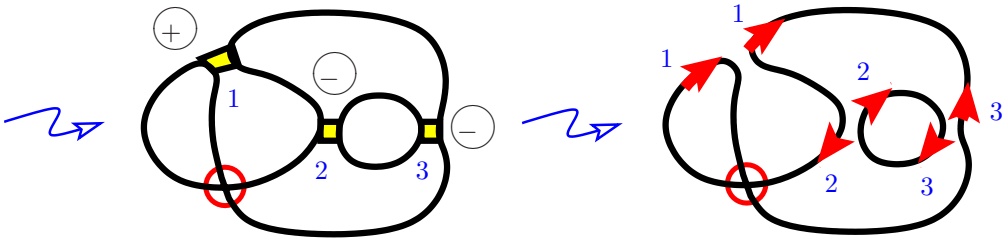
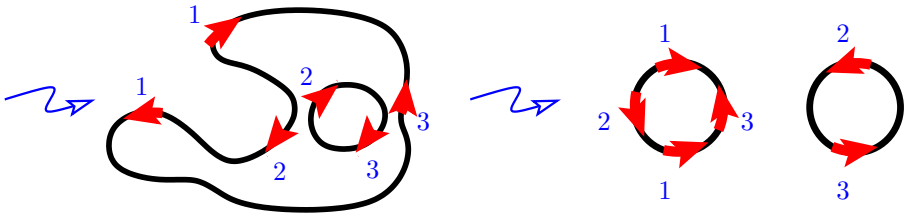


Diagram State s

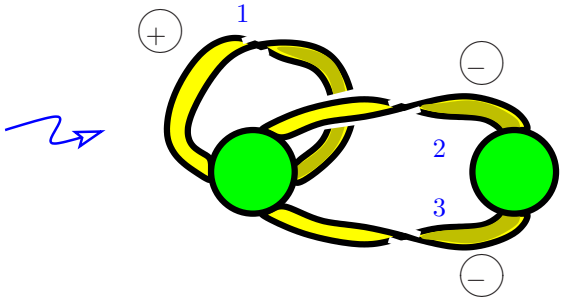


Attaching planar bands Replacing bands by arrows

1 — ; 2 — ; 3 —



Untwisting state circles Pulling state circles apart



Forming the ribbon graph G_L^s

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[Ch] S. Chmutov, *Generalized duality for graphs on surfaces and the signed Bollobás-Riordan polynomial*, Journal of Combinatorial Theory, Ser. B **99**(3) (2009) 617–638; preprint `arXiv:math.CO/0711.3490`.

[ChVo] S. Chmutov, J. Voltz, *Thistlethwaite's theorem for virtual links*. Journal of Knot Theory and Its Ramifications, **17**(10) (2008) 1189-1198; preprint `arXiv:math.GT/0704.1310`.

[DFKLS] O. Dasbach, D. Futer, E. Kalfagianni, X.-S. Lin, N. Stoltzfus, *The Jones polynomial and graphs on surfaces*, Journal of Combinatorial Theory, Ser.B **98** (2008) 384–399; preprint `math.GT/0605571`.

[Ka1] L. H. Kauffman, *New invariants in knot theory*, Amer. Math. Monthly **95** (1988) 195–242.

[Ka2] L. H. Kauffman, *A Tutte polynomial for signed graphs*, Discrete Appl. Math. **25** (1989) 105–127.