Knots and Graphs
Working Group [Summer 2011]
Instructor: ***Sergei Chmutov***

RESEARCH PROJECTS

Project 1. ***Higher dimensional matrix-tree type theorems.*** (Carlos Bajo, Bradley Burdick)

A classical matrix-tree theorem expresses the determinant of some matrix constructed from a graph (principal minor of the Laplacian) as a sum over all spanning trees of the graph. There are generalizations of this theorem to hypergraphs or simplicial complexes [MV, DKM]. Some version of this theorem provides a formula for the first non-zero coefficient of the Conway polynomial of a (virtual) link. In the case of graphs, the generating function of the spanning trees appears as a free term of the multivariable Tutte polynomial. The Tutte polynomial was generalized to simplicial complexes in [KR]. The aim of this project is to relate the free terms of the Krushkal-Renardy polynomial with the simplicial matrix-tree theorem.

Project 2. ***Matrix-quasi-tree theorem for ribbon graphs.*** (Sean Collins, Patrick Schnell)

For ribbon graphs instead of spanning trees it is more natural to consider spanning *quasi-trees* [CKS]. The project is intended to search for an appropriate matrix-tree type theorem for quasi-trees. That is to look for a matrix whose determinant would give the generating function of quasi-trees in a given ribbon graph.

Project 3. ***Matroids and oriented graphs on oriented surfaces*** (Ross Askanazi, Jonathan Michel)

There is a polynomial invariants of graphs embedded into surfaces, the Las Vergnas polynomial [LV1, LV2] coming from matroids. It was shown in [ACEMS] that it is a specialization of the Krushkal polynomial from [Kr]. With an oriented graph on an oriented surface one can associate an oriented matroid. We will try to generalize the Las Vergnas polynomial to oriented matroids and try to relate it with the Krushkal polynomial which generalizes the Tutte polynomial for graphs on surfaces. An oriented version of it might be related to the oriented version of the Tutte polynomial from [GT].

Project 4. ***Planar graphs.*** (Zakariya Bainazarov, Ji Hoon Chun, Andrew Krieger)

The classical Kuratowski theorem states that a graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$ (see, for example, [Har]). Recently, some interest for planarity of graphs with *crossing structure, X-graphs*, appeared in knot theory [Va]. An X-graph is a regular graph each vertex of which has degree 4 and the four edges meeting at a vertex are parted into two pairs of two edges each. An X-embedding of an X-graph is an embedding of the graph to a surface when the partition form a crossing. V. Vassiliev [Va] formulated a conjecture stating that a plane X-embedding of an X-graph exists if and only if it does not contain two circuits intersection at a single vertex. The conjecture was proved in [Man], and simplified in [Sko]. In this project we will try to further simplify the proof of this conjecture, relate it to the other planarity criteria, and look for a general formula for the minimal X-genus of X-graphs.

Project 5. ***Virtual links and arrow polynomial.*** (Robert Ivancic, Benjamin O'Connor)

Virtual link diagrams, besides the *classical crossings* with the information of which strand goes on the top and which one goes on bottom provided, may have also *virtual crossings* where this information is not specified. In such form, virtual links were introduced by L. Kauffman in [Ka]. Independently, at about the same time, virtual links were introduced by M. Goussarov, M. Polyak, O. Viro [GPV] in terms of *Gauss diagrams*. For virtual links there is a generalization of the Jones polynomial, the arrow polynomial [DK], which is defined withing the Kauffman approach. cannot be obtain as a specialization of the relative Tutte polynomial. The goal of the project is to find a Gauss diagram approach to the arrow polynomial.

References

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