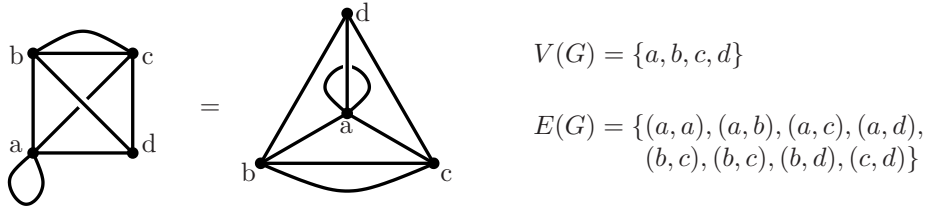


## Graphs

**Definition.** A graph  $G$  is a finite set of vertices  $V(G)$  and a finite set  $E(G)$  of unordered pairs  $(x, y)$  of vertices  $x, y \in V(G)$  called edges.

A graph may have loops  $(x, x)$  and multiple edges when a pair  $(x, y)$  appears in  $E(G)$  several times. Pictorially we represent the vertices by points and edges by lines connecting the corresponding points. Topologically a graph is a 1-dimensional cell complex with  $V(G)$  as the set of 0-cells and  $E(G)$  as the set of 1-cells. Here are two pictures representing the same graph.



## Tutte polynomial

Chromatic polynomial  $C_G(q)$ .

A coloring of  $G$  with  $q$  colors is a map  $c : V(G) \rightarrow \{1, \dots, q\}$ . A coloring  $c$  is proper if for any edge  $e: c(v_1) \neq c(v_2)$ , where  $v_1$  and  $v_2$  are the endpoints of  $e$ .

**Definition 1.**  $C_G(q) := \#$  of proper colorings of  $G$  in  $q$  colors.

**Properties (Definition 2).**

$$C_G = C_{G-e} - C_{G/e} ;$$

$$C_{G_1 \sqcup G_2} = C_{G_1} \cdot C_{G_2}, \quad \text{for a disjoint union } G_1 \sqcup G_2 ;$$

$$C_{\bullet} = q .$$

Tutte polynomial  $T_G(x, y)$ .

**Definition 1.**

$$T_G = T_{G-e} + T_{G/e} \quad \text{if } e \text{ is neither a bridge nor a loop ;}$$

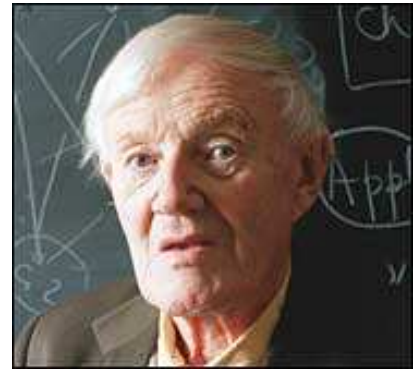
$$T_G = x T_{G/e} \quad \text{if } e \text{ is a bridge ;}$$

$$T_G = y T_{G-e} \quad \text{if } e \text{ is a loop ;}$$

$$T_{G_1 \sqcup G_2} = T_{G_1} \cdot T_{G_2} \quad \text{for a disjoint union } G_1 \sqcup G_2$$

$$\text{and a one-point join } G_1 \cdot G_2 ;$$

$$T_{\bullet} = 1 .$$



**Properties.**

$$T_G(1, 1) \quad \text{is the number of spanning trees of } G ;$$

$$T_G(2, 1) \quad \text{is the number of spanning forests of } G ;$$

$$T_G(1, 2) \quad \text{is the number of spanning connected subgraphs of } G ;$$

$$T_G(2, 2) = 2^{|E(G)|} \quad \text{is the number of spanning subgraphs of } G .$$

$$C_G(q) = q^{k(G)} (-1)^{r(G)} T_G(1 - q, 0) .$$

**Definition 2.**

- Let  $\bullet$   $F$  be a graph;
- $\bullet$   $v(F)$  be the number of its vertices;
  - $\bullet$   $e(F)$  be the number of its edges;
  - $\bullet$   $k(F)$  be the number of components of  $F$ ;

- $r(F) := v(F) - k(F)$  be the *rank* of  $F$ ;
- $n(F) := e(F) - r(F)$  be the *nullity* of  $F$ ;

$$T_G(x, y) := \sum_{F \subseteq E(G)} (x - 1)^{r(G) - r(F)} (y - 1)^{n(F)}$$

### Dichromatic polynomial $Z_G(q, v)$ (Definition 3).

Let  $Col(G)$  denote the set of colorings of  $G$  with  $q$  colors, and let  $D : E(G) \times Col(G) \rightarrow \{0, 1\}$  be defined by the formula  $D(e, c) = 1$  if and only if  $c(v_1) = c(v_2)$ , where  $v_1$  and  $v_2$  are the endpoints of  $e$ .

$$Z_G(q, v) := \sum_{c \in Col(G)} \prod_{e \in E(G)} (1 + vD(e, c))$$

### Properties .

$$\begin{aligned} Z_G &= Z_{G-e} + vZ_{G/e} ; \\ Z_{G_1 \sqcup G_2} &= Z_{G_1} \cdot Z_{G_2} , \quad \text{for a disjoint union } G_1 \sqcup G_2 ; \\ Z_{\bullet} &= q ; \end{aligned}$$

$$Z_G(q, v) = \sum_{F \subseteq E(G)} q^{k(F)} v^{e(F)} ;$$

$$\begin{aligned} C_G(q) &= Z_G(q, -1) ; \\ Z_G(q, v) &= q^{k(G)} v^{r(G)} T_G(1 + qv^{-1}, 1 + v) ; \\ T_G(x, y) &= (x - 1)^{-k(G)} (y - 1)^{-v(G)} Z_G((x - 1)(y - 1), y - 1) . \end{aligned}$$

### Potts model in statistical mechanics (Definition 4).

Potts model (C.Domb 1952);  $q = 2$  the Ising model (W.Lenz, 1920)

Let  $G$  be a graph.

Particles are located at vertices of  $G$ . Each particle has a *spin*, which takes  $q$  different values . A *state*,  $\sigma \in \mathcal{S}$ , is an assignment of spins to all vertices of  $G$ . Neighboring particles interact with each other only if their spins are the same.

The energy of the interaction along an edge  $e$  is  $-J_e$  (*coupling constant*). The model is called *ferromagnetic* if  $J_e > 0$  and *antiferromagnetic* if  $J_e < 0$ .

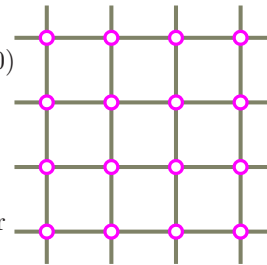
Energy of a state  $\sigma$  (*Hamiltonian*),

$$H(\sigma) = - \sum_{(a,b)=e \in E(G)} J_e \delta(\sigma(a), \sigma(b)).$$

*Boltzmann weight* of  $\sigma$ :

$$e^{-\beta H(\sigma)} = \prod_{(a,b)=e \in E(G)} e^{J_e \beta \delta(\sigma(a), \sigma(b))} = \prod_{(a,b)=e \in E(G)} \left( 1 + (e^{J_e \beta} - 1) \delta(\sigma(a), \sigma(b)) \right),$$

where the *inverse temperature*  $\beta = \frac{1}{\kappa T}$ ,  $T$  is the temperature,  $\kappa = 1.38 \times 10^{-23}$  joules/Kelvin is the *Boltzmann constant*.



The Potts partition function (for  $x_e := e^{J_e\beta} - 1$ )

$$Z_G(q, x_e) := \sum_{\sigma \in \mathcal{S}} e^{-\beta H(\sigma)} = \sum_{\sigma \in \mathcal{S}} \prod_{e \in E(G)} (1 + x_e \delta(\sigma(a), \sigma(b)))$$

**Properties of the Potts model** Probability of a state  $\sigma$ :  $P(\sigma) := e^{-\beta H(\sigma)} / Z_G$ .  
Expected value of a function  $f(\sigma)$ :

$$\langle f \rangle := \sum_{\sigma} f(\sigma) P(\sigma) = \sum_{\sigma} f(\sigma) e^{-\beta H(\sigma)} / Z_G .$$

Expected energy:  $\langle H \rangle = \sum_{\sigma} H(\sigma) e^{-\beta H(\sigma)} / Z_G = -\frac{d}{d\beta} \ln Z_G$ .

Fortuin—Kasteleyn'1972:  $Z_G(q, x_e) = \sum_{F \subseteq E(G)} q^{k(F)} \prod_{e \in F} x_e$ ,

where  $k(F)$  is the number of connected components of the spanning subgraph  $F$ .  
 $Z_G = Z_{G \setminus e} + x_e Z_{G/e}$ .

**Spanning tree generating function (Definition 5).**

For a connected graph  $G$  fix an order of its edges:  $e_1, e_2, \dots, e_m$ . Let  $T$  be a spanning tree.

An edge  $e_i \in E(T)$  is called *internally active (live)* if  $i < j$  for any edge  $e_j$  connecting the two components of  $T - e_i$ .

An edge  $e_j \notin E(T)$  is called *externally active (live)* if  $j < i$  for any edge  $e_i$  in the unique cycle of  $T \cup e_j$ .

Let  $i(T)$  and  $j(T)$  be the numbers of internally and externally active edges correspondingly.

$$T_G(x, y) := \sum_T x^{i(T)} y^{j(T)}$$