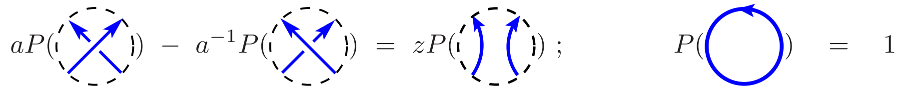
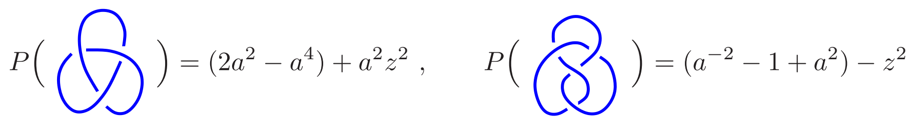
HOMFLYPT polynomial

The *HOMFLY polynomial* $P(L)$ is defined as a Laurent polynomial in two variables a and z with integer coefficients satisfying the following skein relation and the initial condition:



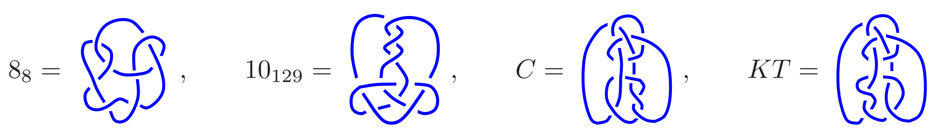
The existence of such an invariant is a difficult theorem. It was established simultaneously and independently by five groups of authors [HOM, PT] (see also [Lik]). The HOMFLY polynomial is equivalent to the collection of quantum invariants associated with the Lie algebra $\mathfrak{sl}_N$ and its standard N-dimensional representation for all values of N.

Examples.



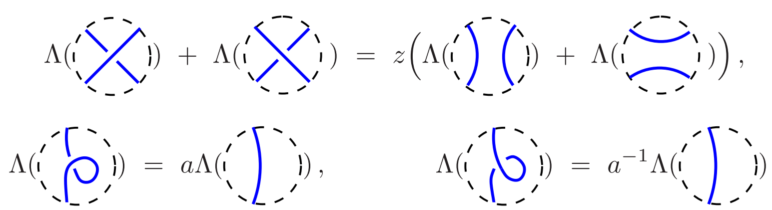
Properties.

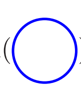
(1) HOMFLYPT polynomial of a knot is preserved when the knot orientation is reversed.
(2) $P(\overline{L}) = \overline{P(L)}$, where $\overline{L}$ is the mirror reflection of L and $\overline{P(L)}$ is the polynomial obtained from $P(L)$ by substituting a^{-1} for a;
(3) $P(K_1 \# K_2) = P(K_1) \cdot P(K_2)$;
(4) $P(L_1 \sqcup L_2) = \frac{a-a^{-1}}{z} \cdot P(L_1) \cdot P(L_2)$;
(5) $P(8_8) = P(10_{129})$ and $P(C) = P(KT)$ for the Conway, C, and the Kinoshita–Terasaka, KT, knots below.



Two-variable Kauffman polynomial

L. Kauffman [Ka] found another invariant Laurent polynomial $F(L)$ in two variables a and z. Firstly, for a unoriented link diagram D we define a polynomial $\Lambda(D)$ which is invariant under Reidemeister moves II and III and satisfies the skein relations



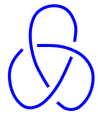
and the initial condition $\Lambda($  $) = 1$.

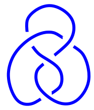
Now, for any diagram D of an oriented link L we put

$$F(L) := a^{-w(D)}\Lambda(D).$$

The Kauffman polynomial is equivalent to the collection of the quantum invariants associated with the Lie algebra $\mathfrak{so}_N$ and its standard N-dimensional representation for all values of N.

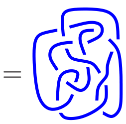
Examples.

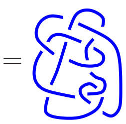
$$F\Big($$  $$\Big) = (-2a^2 - a^4) + (a^3 + a^5)z + (a^2 + a^4)z^2 \ ,$$

$$F\Big($$  $$\Big) = (-a^{-2} - 1 - a^2) + (-a^{-1} - a)z + (a^{-2} + 2 + a^2)z^2 + (a^{-1} + a)z^3 \ .$$

Properties.

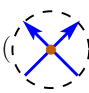
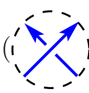
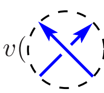
(1) $F(K)$ is preserved when the knot orientation is reversed.
(2) $F(\overline{L}) = \overline{F(L)}$, where $\overline{L}$ is the mirror reflection of L, and $\overline{F(L)}$ is the polynomial obtained from $F(L)$ by substituting a^{-1} for a;
(3) $F(K_1 \# K_2) = F(K_1) \cdot F(K_2)$;
(4) $F(L_1 \sqcup L_2) = \Big((a + a^{-1})z^{-1} - 1\Big) \cdot F(L_1) \cdot F(L_2)$;
(5) $F(11_{255}) = F(11_{257})$;
(these knots can be distinguished by the Conway and, hence, by the HOMFLY polynomial).

$11_{255} =$ 

$11_{257} =$ 

Vassiliev knot invariants

The main idea of the combinatorial approach to the theory of *Vassiliev knot invariants*, also known as *finite type invariants*, is to extend a knot invariant v to singular knots with double points according to the following rule, which we will refer to as *Vassiliev skein relation*:

$$v\Big($$  $$\Big) := v\Big($$  $$\Big) - v\Big($$  $$\Big) \ .$$

Definition. A knot invariant is said to be a *Vassiliev invariant* of order (or degree) $\leq n$ if its extension vanishes on all singular knots with more than n double points.

Denote by $\mathcal{V}_n$ the set of Vassiliev invariants of order $\leq n$ with values in the field of complex numbers $\mathbb{C}$. The definition implies that, for each n, the set $\mathcal{V}_n$ forms a complex vector space. Moreover, $\mathcal{V}_n \subseteq \mathcal{V}_{n+1}$, so we have an increasing filtration

$$\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots \subseteq \mathcal{V}_n \subseteq \cdots \subseteq \mathcal{V} := \bigcup_{n=0}^{\infty} \mathcal{V}_n \ .$$

It will be shown that the spaces $\mathcal{V}_n$ have finite dimension, and that the quotients $\mathcal{V}_n/\mathcal{V}_{n-1}$ admit a nice combinatorial description. The study of these spaces is the main purpose of the combinatorial Vassiliev invariant theory. The exact dimension of $\mathcal{V}_n$ is known only for $n \leq 12$:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{V}_n$	1	1	2	3	6	10	19	33	60	104	184	316	548
$\dim \mathcal{V}_n/\mathcal{V}_{n-1}$	1	0	1	1	3	4	9	14	27	44	80	132	232

References

[HOM] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, A. Ocneanu, *A new polynomial invariant of knots and links*, Bull. AMS **12** (1985) 239–246.

[Ka] L. Kauffman, *An invariant of regular isotopy*, Transactions of the AMS, **318**, no. 2, 417–471, 1990.

[Lik] W. B. R. Lickorish, *An introduction to knot theory*, Springer-Verlag New York, Inc. (1997).

[PT] J. Przytycki, P. Traczyk, *Invariants of links of the Conway type*, Kobe J. Math. **4** (1988) 115–139.