Dual representable matroids

Let $E = \{v_1, \ldots, v_n\}$ be a collection of vectors in a vector space U and M be a matroid of their linear dependences. Consider an n-dimensional vector space V with a basis $e_1, \ldots, e_n$ and a linear map $f : V \to U$ sending e_k to v_k. Denote the kernel of this map by W. It is a subspace of V and there is a natural inclusion map $i : W \hookrightarrow V$. There is the dual map $W^* \xleftarrow{i^*} V^*$ of dual vector spaces. The space V^* has a natural dual basis $e_1^*, \ldots, e_n^*$. Their images $i^*(e_1^*), \ldots, i^*(e_n^*)$ is a collection of vectors in the space W^*. These vectors with the structure of linear dependences between them form the dual matroid M^*.

The Las Vergnas polynomial

Reference: M. Las Vergnas [LV].

Matroid perspectives.

A bijection $M \to M'$ is called *matroid perspective* if any circuit of M is mapped to a union of circuites of M'. Equivalently,

$$r_M(X) - r_M(Y) \geqslant r_{M'}(X) - r_{M'}(Y) \qquad \text{for all} \quad Y \subseteq X.$$

Example.

For graphs G and G^* dually embedded in a surface, then the map of the bond matroid of G^* onto the circuit matroid of G, $\mathcal{B}(G^*) \to \mathcal{C}(G)$, is a matroid perspective.

Definition.

$$T_{M\to M'}(x,y,z) \ := \sum_{X\subseteq M} (x-1)^{r(M')-r_{M'}(X)}(y-1)^{n_M(X)} z^{(r(M)-r_M(X))-(r(M')-r_{M'}(X))}$$

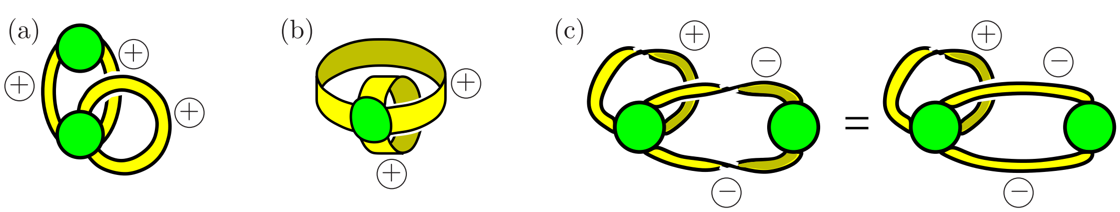
Properties.

$$T_M(x,y) = T_{M\to M}(x,y,z) \ ;$$
$$T_M(x,y) = T_{M\to M'}(x,y,x-1) \ ;$$
$$T_{M'}(x,y) = (y-1)^{r(M)-r(M')} T_{M\to M'}(x,y,\tfrac{1}{y-1}) \ ;$$

Ribbon graphs (graphs on surfaces)

Definition. A *ribbon graph* G is a surface (possibly non-orientable) with boundary, represented as the union of two sets of closed topological discs called *vertices* $V(G)$ and *edges* $E(G)$, satisfying the following conditions:

- these vertices and edges intersect by disjoint line segments;
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- every edge contains exactly two such line segments.



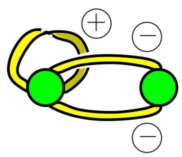
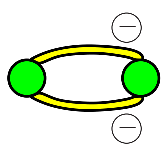
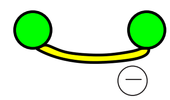
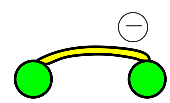
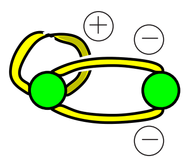
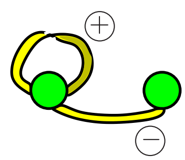
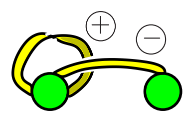
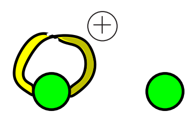
The Bollobás-Riordan polynomial

Reference: B. Bollobás and O. Riordan [BR].

$$R_G(\{x_e, y_e\}, X, Y, Z) \;:= \sum_{F \subseteq G} \Big(\prod_{e \in F} x_e\Big)\Big(\prod_{e \in \overline{F}} y_e\Big) X^{r(G)-r(F)} Y^{n(F)} Z^{k(F)-\mathrm{bc}(F)+n(F)}$$

For signed graphs, we set $\begin{cases} x_+ = 1, & x_- = (X/Y)^{1/2}, \\ y_+ = 1, & y_- = (Y/X)^{1/2}. \end{cases}$

Example.

				
(k,r,n,bc) term of R_G	(1,1,1,2) X	(1,1,0,1) 1	(1,1,0,1) 1	(2,0,0,2) Y
				
	(1,1,2,1) XYZ^2	(1,1,1,1) YZ	(1,1,1,1) YZ	(2,0,1,2) Y^2Z

$$R_G(X, Y, Z) = X + 2 + Y + XYZ^2 + 2YZ + Y^2Z$$

Properties.

$R_G = x_e R_{G/e} + y_e R_{G-e}$ if e is ordinary, that is neither a bridge nor a loop,

$R_G = (x_e + X y_e) R_{G/e}$ if e is a bridge.

$R_{G_1 \sqcup G_2} = R_{G_1 \cdot G_2} = R_{G_1} \cdot R_{G_2}$

References

[BR] B. Bollobás and O. Riordan, *A polynomial of graphs on surfaces*, Math. Ann. **323** (2002) 81–96.

[LV] M. Las Vergnas, *On the Tutte polynomial of a morphism of matroids*, Annals of Discrete Mathematics **8** (1980) 7–20.