# <u>Knots and Graphs</u>

## Working Group [Summer 2012]

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## <u>RESEARCH PROJECTS</u>

**Project 1.** ***Virtual links and arrow polynomial.*** (<u>Ben O'Connor</u>, Erin Zwick, Noah Taylor)

Virtual link diagrams, besides the *classical crossings* with the information of which strand goes on the top and which one goes on bottom provided, may have also *virtual crossings* where this information is not specified. In such form, virtual links were introduced by L. Kauffman in [Ka]. Independently, at about the same time, virtual links were introduced by M. Goussarov, M. Polyak, O. Viro [GPV] in terms of *Gauss diagrams*. For virtual links there is a generalization of the Jones polynomial, the arrow polynomial [DK], which is defined withing the Kauffman approach. L.Kauffman announced [Ka1] a more general invariant of virtual knots taking values in the set of graphs. Last year Ben O'Connor found a gap in this work. The goal of the project is to try to fill in the gap and/or to try to generalize the arrow polynomial.

**Project 2.** ***Higher dimensional Tutte polynomial.*** (<u>Jon Michel</u>, Joel Arter, Kailin Huang, Nick Kosar)

The higher dimensional Tutte polynomial is an invariant for cell complexes introduced in [KR]. It was studied last year by Carlos Bajo, Bradley Burdick [BBC]. In the classical situation for graphs the Tutte polynomial specializes to the so called *flow polynomial* (see, for example [Bo]). Flow polynomial for simplicial complexes was introduced recently in [BK]. One goal of this project is to relate their flow polynomial to the Tutte-Krushkal-Renardy polynomial.

The another goal of this project is to understand the relation of the modified Tutte-Krushkal-Renardy polynomial from [BBC] with the Tutte polynomial of arithmetic matroids from [DAM1] and their arithmetic flow polynomial from [DAM2].

**Project 3.** ***Matrix-quasi-tree theorem for ribbon graphs.*** (<u>Andrew Krieger</u>, Isaac Smith, Amy Weisman)

This is a continuation of the project from the last summer.

A classical matrix-tree theorem expresses the determinant of some matrix constructed from a graph (principal minor of the Laplacian) as a sum over all spanning trees of the graph. For ribbon graphs instead of spanning trees it is more natural to consider spanning *quasi-trees* [CKS]. The project is intended to search for an appropriate matrix-tree type theorem for quasi-trees. That is to look for a matrix whose determinant would give the generating function of quasi-trees in a given ribbon graph. Some nice results in this direction were obtained by Patrick Schnell last summer. They need to be cleaned up and related to article [DFKLS].

**Project 4.** ***Planar graphs.*** (Ji Hoon Chun, Robin Baidya, Tyler Friesen, Peter Tian)

This is also a continuation of the project from the last summer.

The classical Kuratowski theorem states that a graph is planar if and only if it has no subgraph homeomorphic to $K_5$ or $K_{3,3}$ (see, for example, [Har]). Recently, some interest for planarity of graphs with *crossing structure, $X$-graphs*, appeared in knot theory [Va]. An $X$-graph is a regular graph each vertex of which has degree 4 and the four edges meeting at a vertex are parted into two pairs of two edges each. An $X$-embedding of an $X$-graph is an embedding of the graph to a surface when the partition form a crossing. V. Vassiliev [Va] formulated a conjecture stating that a plane $X$-embedding of an $X$-graph exists if and only if it does not contain two circuits intersection at a single vertex. The conjecture was proved in [Man1]. In this project we will try to simplify the proof of this conjecture, relate it to the other planarity criteria, and study a general formula for the minimal $X$-genus of $X$-graphs from [Man2].

# References


[BBC] C. Bajo, B. Burdick, S.Chmutov, *On the Tutte-Krushkal-Renardy polynomial for cell complexes*, Preprint `arXiv:1204.3563v1 [math.CO]`.

[BK] M. Beck, Y. Kemper *Flows on Simplicial Complexes*, Preprint `arXiv:1206.2260v1 [math.CO]`.

[Bo] B. Bollobás, *Modern graph theory*, Graduate Texts in Mathematics **184**, Springer, New York, 1998.

[CKS] A. Champanerkar, I. Kofman, N. Stoltzfus, *Quasi-tree expansion for the BollobásRiordanTutte polynomial*, Bulletin of the London Mathematical Society **43**(5) (2011) 972–984. Preprint `arXiv:0705.3458 [math.CO]`.

[DAM1] M. D'Adderio, L. Moci, *Arithmetic matroids, Tutte polynomial, and toric arrangements*, Preprint `arXiv:1105.3220v3 [math.CO]`.

[DAM2] M. D'Adderio, L. Moci, *Graph colorings, flows and arithmetic Tutte polynomial*, Preprint `arXiv:1108.5537v1 [math.CO]`.

[DFKLS] O. Dasbach, D. Futer, E. Kalfagianni, X.-S. Lin, N. Stoltzfus, *Alternating sum formulae for the determinant and other link invariants*, Journal of Knot Theory and Its Ramifications, **19**(6) (2010) 765–782. Preprint `arXiv:math/0611025v2 [math.GT]`.

[DK] H. Dye, L. Kauffman, *Virtual Crossing Number and the Arrow Polynomial*, Preprint `arXiv:math.GT/0810.3858`.

[GPV] M. Goussarov, M. Polyak, O. Viro, *Finite type invariants of classical and virtual knots*, Topology, **39**, 1045–1068 (2000).

[Har] F. Harary, *Graph Theory*, Addison-Wesley, 1969.

[Ka] L. Kauffman, *Virtual knot theory*, European Journal of Combinatorics, **20** (1999) 663–690. Preprint `arXiv:math/9811028v3 [math.GT]`.

[Ka1] L. Kauffman, *An Extended Bracket Polynomial for Virtual Knots and Links*, Journal of Knot Theory and Its Ramifications, **18**(10) (2009) 1369–1422. Preprint `arXiv:0712.2546v5 [math.GT]`.

[KR] V. Krushkal, D. Renardy, *A polynomial invariant and duality for triangulations*, Preprint `arXiv:1012.1310v1 [math.CO]`.

[Man1] V. O. Manturov, *A proof of Vassilievs conjecture on the planarity of singular links*, Izvestiya: Mathematics **69**(5) (2005) 1025-1033.

[Man2] V. O. Manturov, *Embeddings of four-valent framed graphs into two-surfaces*, Preprint `arXiv:0804.4245v1 [math.CO]`.

[Va] V. A. Vassiliev, *First-order invariants and cohomology of spaces of embeddings of self-intersecting curves in* $\mathbb{R}^n$, Izvestiya: Mathematics, **69**(5) (2005) 865-912.