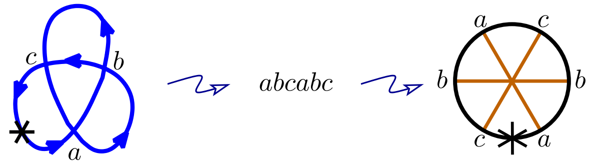
Gauss diagrams.

A *Gauss word* is a word where each letter appears twice considering up to a cyclic permutation. For example, here are equal Gauss words.

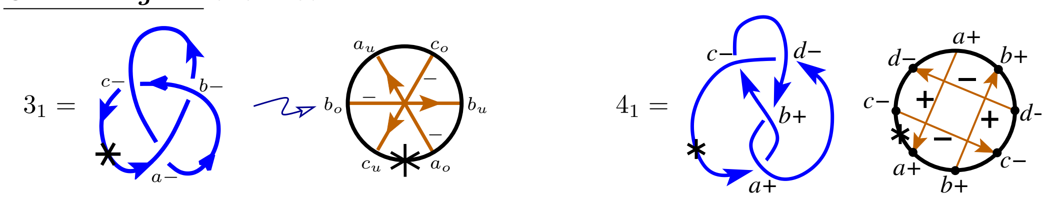
$$abcabc = bcabca = cabcab \ .$$

Gauss word of a plane immersed curve and its *Gauss diagram*:

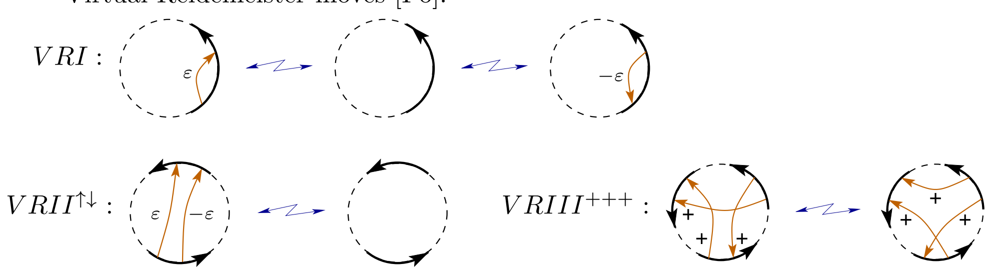


Non-realizable Gauss word: *abab*.

Gauss diagram of a knot:

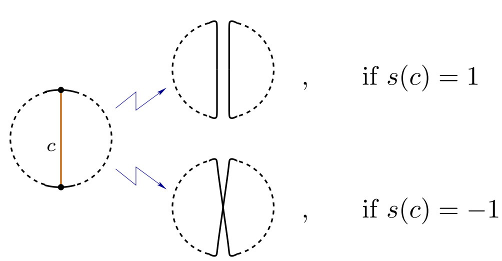


Virtual Reidemeister moves [Po].



Jones polynomial via Gauss diagrams. [Zul]

Let G be a Gauss diagram representing a knot K. Denote by $[G]$ the set of arrows of G. The sign of an arrow $c \in [G]$ can be considered as a value of the function sign : $[G] \to \{-1, +1\}$. A *state* s for G is an arbitrary function $s : [G] \to \{-1, +1\}$; in particular, for a Gauss diagram with n arrows there are 2^n states. The function sign$(\cdot)$ is one of them. With each state s we associate an immersed plane curve in the following way. Double every chord c according to the rule:



Let $|s|$ denote the number of connected components of the curve obtained by doubling all the chords of G. Also, for a state s we define an integer

$$p(s) := \sum_{c\in[G]} s(c)\cdot \mathrm{sign}(c)\,.$$

The defining relations for the Kauffman bracket ($A = t^{-1/4}$, $B = t^{1/4}$, $d = -t^{1/2} - t^{-1/2}$) lead to the following expression for the Jones polynomial.

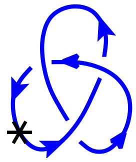
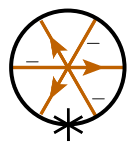
Theorem.

$$J(K) = (-1)^{w(K)} t^{3w(K)/4} \sum_s t^{-p(s)/4} \big(-t^{-1/2} - t^{1/2}\big)^{|s|-1}\ ,$$

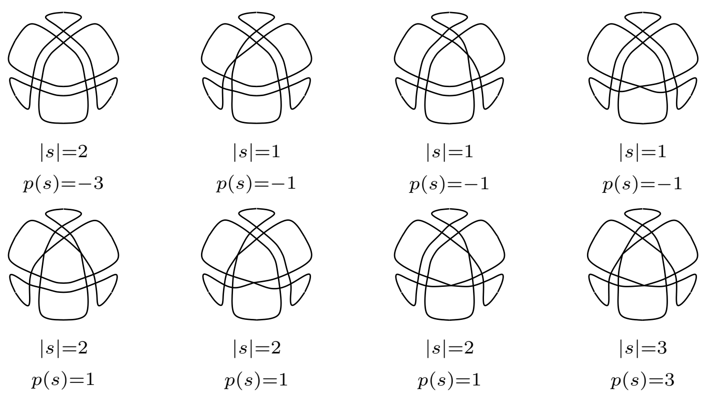
where the sum is taken over all 2^n *states for* G *and* $w(K) = \sum_{c\in[G]} \mathrm{sign}(c)$ *is the writhe of* K.

This formula can be used to extend the Jones polynomial to virtual knots.

Example. For the left trefoil knot 3_1 we have the following Gauss diagram.

$3_1 =$  $G =$  $w(G) = -3$

There are eight states for such a diagram. Here are the corresponding curves and numbers $|s|$, $p(s)$.



Therefore,

$$\begin{aligned} J(3_1) &= -t^{-9/4}\Big(t^{3/4}\big(-t^{-1/2} - t^{1/2}\big) + 3t^{1/4} + 3t^{-1/4}\big(-t^{-1/2} - t^{1/2}\big) \\ &\qquad + t^{-3/4}\big(-t^{-1/2} - t^{1/2}\big)^2\Big) \\ &= -t^{-9/4}\Big(-t^{1/4} - t^{5/4} - 3t^{-3/4} + t^{-3/4}\big(t^{-1} + 2 + t\big)\Big) \\ &= t^{-1} + t^{-3} - t^{-4}\ , \end{aligned}$$

References

[Po] M. Polyak, *Minimal sets of Reidemeister moves*, Quantum Topology **1** (2010) 399–411.

[Zul] L. Zulli, *A matrix for computing the Jones polynomial of a knot*, Topology **34** (1995) 717–729.