A Quantum Gauss-Bonnet Theorem

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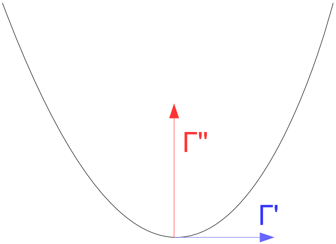
Curvature in the plane

- Let Γ be a smooth curve with orientation in $\mathbb{R}^2$, parametrized by arc length.
- The *curvature* k of Γ is $\pm\|\Gamma''\|$, where the sign is positive if Γ'' is counterclockwise of Γ', and negative if Γ'' is clockwise of Γ'.
- Curvature measures the change in direction per unit distance along the curve.



Hopf's Umlaufsatz

- A curve is *simple* if it has no self-intersections.
- Hopf's Umlaufsatz: If Γ is a simple smooth closed curve,

$$\int_{S^1} k(s)\, ds = \pm 2\pi$$

- Umlaufsatz with corners: If Γ is a simple piecewise-smooth closed curve, let C be the set of corners of Γ and for each $c \in C$, let φ_c be the exterior angle at c. Then

$$\int_{S^1} k(s)\, ds + \sum_{c \in C} \varphi_c = \pm 2\pi$$

The Whitney-Graustein Theorem

- Two smooth curves are said to be *regular homotopic* to each other if one can be continuously deformed into the other such that at any moment in time, the intermediate curve is a smooth curve.
- The *rotation number* of a smooth closed curve is given by the formula

$$\frac{1}{2\pi}\int_{S^1} k(s)\,ds$$

  It is always an integer, and this formula can be viewed as the Umlaufsatz with multiplicities.
- Whitney-Graustein Theorem: Two smooth closed curves in the plane are regular homotopic if and only if they have the same rotation number.

The J^+ invariant

- A curve is *generic* if its only self-intersections are transverse double points.
- Theorem: If two generic smooth closed curves in any surface are regular homotopic, then one can be deformed into the other by diffeomorphism of the surface and a finite number of self-tangency moves and triple-point moves.
- The J^+ invariant associates an integer to each generic smooth curve in a given orientable surface, such that this integer changes by 2 at direct self-tangency moves and is unchanged by opposite self-tangency moves and by triple-point moves.
- This almost determines J^+ uniquely; we just need to specify its value on a representative of each regular homotopy class. There is a standard such specification for planar curves.

The J^+ invariant (continued)

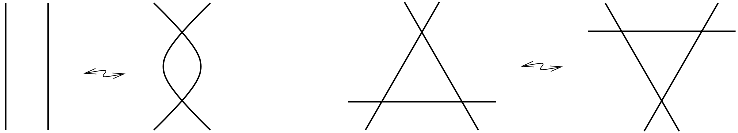


Figure: Self-tangency and triple-point moves

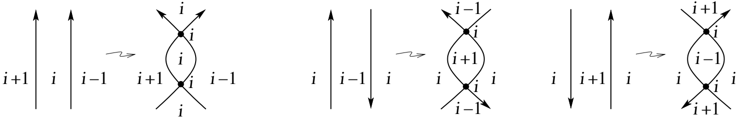


Figure: A direct self-tangency move and two opposite self-tangency moves

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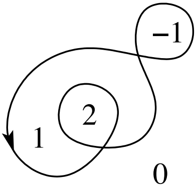
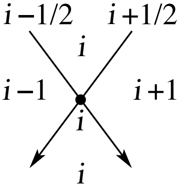
Images due to Lanzat and Polyak

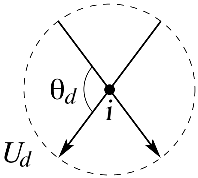
Winding numbers

- Given a curve Γ in the plane and a point p not in Γ, we define the *winding number* or *index* $\mathrm{ind}_\Gamma(p)$ to be the total number of (signed) turns made by Γ around p.
- The winding number changes by ± 1 when p crosses over Γ, according to the orientation of the section of Γ it crosses.
- Given Γ, the winding number ind_Γ thus gives a function on $\mathbb{R}^2 \setminus \Gamma$; we extend this to all of $\mathbb{R}^2$ by defining $\mathrm{ind}_\Gamma(p)$ for $p \in \Gamma$ to be the average of the winding numbers of the regions of $\mathbb{R}^2 \setminus \Gamma$ in a neighborhood of p.

Figures







Images due to Lanzat and Polyak

Lanzat and Polyak's Polynomial Invariant

- Let Γ be a generic smooth closed curve in the plane. Let X be its set of double points, and for each $d = \Gamma(t_1) = \Gamma(t_2) \in X$, let θ_d be the (non-oriented) angle between $\Gamma'(t_1)$ and $-\Gamma'(t_2)$.
- Then Lanzat and Polyak define an associated "quantum invariant" $I_q(\Gamma) \in \mathbb{R}[q^{1/2}, q^{-1/2}]$ as follows:

$$\frac{1}{2\pi}\left(\int_{S_1} k(t) \cdot q^{\mathrm{ind}_\Gamma(\Gamma(t))}\,dt - \sum_{d\in X} \theta_d \cdot q^{\mathrm{ind}_\Gamma(d)}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})\right)$$

- They showed using the Hopf Umlaufsatz with corners that the expression is invariant under planar isotopy.

Its relation to the rotation number and J^+

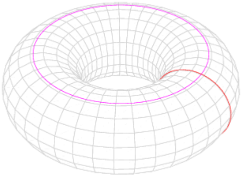
- Substituting $q = 1$ into Lanzat and Polyak's polynomial gives
$$\frac{1}{2\pi}\int_{S^1} k(s)\,ds,$$
the rotation number. Hence we say it is a *quantum deformation* of the rotation number (or of the Umlaufsatz).
- Lanzat and Polyak showed that the first derivative of their polynomial at $q = 1$, $I_1'(\Gamma)$, changes by -1 under direct self-tangencies and is invariant under opposite self-tangencies and triple-point modifications, so $-2I_1'(\Gamma)$ changes by 2 at direct self-tangencies and is invariant under opposite self-tangencies and triple-point moves.
- Thus $J^+(\Gamma) = -2I_1'(\Gamma)$ up to addition of some constant depending only on the regular homotopy class of Γ.

Problems with generalizing to curves in surfaces

- What takes the role of the winding number in the definition of the integral?
- What replaces the Umlaufsatz in the proof?
- Lanzat and Polyak's polynomial is a quantum deformation of the formula for rotation number; what should the generalization be a deformation of?

Homologically trivial curves

- The most important facts about the winding number are that it is locally constant on $S \setminus \Gamma$ and it changes by the appropriate amount when crossing over Γ.
- In some cases, there is no function on $S \setminus \Gamma$ satisfying this property:



- When there is, we say Γ is *homologically trivial*.
- Given a homologically trivial curve Γ in a connected oriented surface S and $b \in S \setminus \Gamma$, let $\mathrm{ind}_{\Gamma,b} : S \setminus \Gamma \to \mathbb{Z}$ be the unique locally constant function which sends b to 0 and changes by the appropriate amount when crossing over Γ.

Geodesic Curvature

- To generalize the Umlaufsatz to surfaces we need a concept of curvature.
- In order to talk about curvature we need to have a concept of lengths and angles.
- The following concepts will apply to any surface with a Riemannian metric, but I will describe them for the less general case of a surface embedded in $\mathbb{R}^3$.
- As with planar curvature, we parametrize Γ by arc length, but where planar curvature is the signed magnitude of Γ'', *geodesic curvature* is the signed magnitude of the projection of Γ'' onto the tangent plane T_pS.
- Examples: The geodesic curvature of a curve in the plane is its planar curvature. The geodesic curvature of a great circle on a sphere is constantly zero.

The Gauss-Bonnet Theorem

- Gauss-Bonnet Theorem: If S is a closed subset of a surface and ∂S is piecewise smooth with finite set C of corners, then

$$\chi(S) = \frac{1}{2\pi}\left(\int_S K\,dA + \int_{\partial S} k_g\,ds + \sum_{c\in C}\varphi_c\right)$$

- If S is the entire surface (without boundary) then this reduces to the Gauss-Bonnet theorem presented earlier by Dr. Farb.
- If S is a subset of the plane and S is homeomorphic to a disk, then this reduces to Hopf's Umlaufsatz with corners.
- More generally, if S is homeomorphic to a disk, the Gauss-Bonnet theorem is like the Umlaufsatz with a corrective term for Gaussian curvature.

Rotation numbers of homologically trivial curves

Given an oriented surface S, the *rotation number* is the unique way of assigning a value in $\mathbb{Z}/\chi(S)\mathbb{Z}$ (or $\mathbb{Z}$ if $\chi(S) = 0$) to each homologically trivial curve in S such that

1. The rotation number is invariant under regular homotopies.
2. The rotation number of a small counterclockwise curve is 1.
3. The rotation number of the composition of two curves is the sum of their rotation numbers.

McIntyre and Cairns's formula for the rotation number

- Pick a base point b in $S \setminus \Gamma$.
- For $j \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, let S_j be the region of S on which $\mathrm{ind}_{\Gamma,b}$ is greater than j.
- For $j \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, let

$$a_j = \begin{cases} \chi(S_j) - \chi(S) & j < 0 \\ \chi(S_j) & j > 0 \end{cases}$$

- Then the winding number is given by

$$\sum_{j \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}} a_j \mod |\chi(S)|$$

The Gauss-Bonnet Theorem with multiplicities

- Let Γ be a homologically trivial generic smooth curve in a connected closed surface S with Riemannian metric and orientation.
- We can calculate the Euler characteristics in McIntyre and Cairns's formula using the Gauss-Bonnet Theorem; this gives the following formula for the rotation number:

$$\frac{1}{2\pi}\left(\int_{S^1} k_g(t)\,dt + \iint_S K \cdot \mathrm{ind}_{\Gamma,b}\,dA\right) \mod |\chi(S)|$$

- This can be viewed as the Gauss-Bonnet theorem with multiplicities.
- Note that if we change b so that $\mathrm{ind}_{\Gamma,b}$ increases by 1 everywhere, the expression before taking the modulus changes by $\chi(S)$.

The Quantum Gauss-Bonnet Theorem

- Instead of taking $\sum_{j\in\frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}} a_j$, take $\sum_{j\in\frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}} a_j q^j$.
- Again applying the Gauss-Bonnet Theorem to calculate the a_j's, we get

$$\frac{1}{2\pi}\left(\int_{S^1} k_g(t)\cdot q^{\mathrm{ind}_{\Gamma,b}(\Gamma(t))}\,dt\right.$$

$$\left.+\sum_{d\in X}(\pi-\theta_d)q^{\mathrm{ind}_{\Gamma,b}(d)}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})+\iint_S K\cdot\frac{q^{\mathrm{ind}_{\Gamma,b}}-1}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\,dA\right)$$

- This is a topological invariant and a quantum deformation of the rotation number, but it isn't quite a generalization of Lanzat and Polyak's formula.

The Quantum Gauss-Bonnet Theorem (continued)

- The expression

$$\frac{1}{2}\sum_{d\in X}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})q^{\mathrm{ind}_{\Gamma,b}}$$

  is a topological invariant and is equal to 0 at $q=1$, so subtracting it away from the expression on the previous page will still give a topological invariant and deformation of the rotation number.

- Here it is:

$$I_q(\Gamma,b):=\frac{1}{2\pi}\left(\int_{S^1}k_g(t)\cdot q^{\mathrm{ind}_{\Gamma,b}(\Gamma(t))}\,dt\right.$$

$$\left.-\sum_{d\in X}\theta_d\cdot q^{\mathrm{ind}_{\Gamma,b}(d)}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})+\iint_S K\cdot\frac{q^{\mathrm{ind}_{\Gamma,b}}-1}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\,dA\right)$$

Its relation to J^+

- $I_q(\Gamma, b)$ changes the same way under self-tangency and triple-point moves that Lanzat and Polyak's polynomial does, so we might think that $-2I_1'(\Gamma, b)$ gives us $J^+(\Gamma)$ (up to a constant depending on the regular homotopy class of Γ).
- However, $I_1'(\Gamma, b)$ is not invariant under a change of base point b.
- $I_1(\Gamma, b)$ (the formula for rotation number, before taking the modulus) can be used to produce a corrective term to give a formula which doesn't change under change of base point.
- When $\chi(S) \neq 0$

$$J^+(\Gamma) = \frac{I_1(\Gamma, b)^2}{\chi(S)} - 2I_1'(\Gamma, b)$$

up to a constant depending on the regular homotopy class of Γ.

An explicit formula for J^+

$$\frac{1}{4\pi^2\chi(S)}\left(\int_{S^1} k_g(t)\,dt + \iint_S \mathrm{ind}_{\Gamma,b}\,dA\right)^2$$

$$-\frac{1}{\pi}\left(\int_{S^1} k_g(t)\cdot \mathrm{ind}_{\Gamma,b}(\Gamma(t))\,dt - \sum_{d\in X}\theta_d + \frac{1}{2}\iint_S K\cdot(\mathrm{ind}_{\Gamma,b})^2\,dA\right)$$