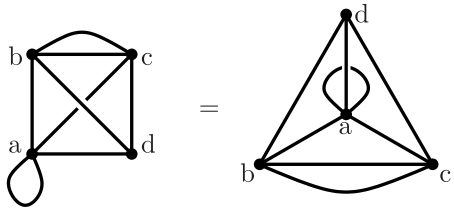
Graphs

Definition. *A graph G is a finite set of vertices $V(G)$ and a finite set $E(G)$ of unordered pairs (x, y) of vertices $x, y \in V(G)$ called edges.*

A graph may have *loops* (x, x) and *multiple edges* when a pair (x, y) appears in $E(G)$ several times. Pictorially we represent the vertices by points and edges by lines connecting the corresponding points. Topologically a graph is a 1-dimensional *cell complex* with $V(G)$ as the set of 0-cells and $E(G)$ as the set of 1-cells. Here are two pictures representing the same graph.



$$V(G) = \{a, b, c, d\}$$

$$E(G) = \{(a, a), (a, b), (a, c), (a, d), (b, c), (b, c), (b, d), (c, d)\}$$

Tutte polynomial

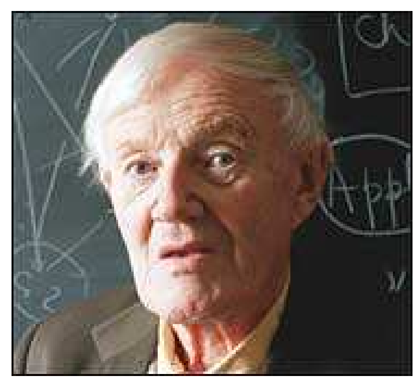
Chromatic polynomial $C_G(q)$.

A *coloring* of G with q colors is a map $c : V(G) \to \{1, \ldots, q\}$. A coloring c is *proper* if for any edge e: $c(v_1) \neq c(v_2)$, where v_1 and v_2 are the endpoints of e.

Definition 1. *$C_G(q) :=$ # of proper colorings of G in q colors.*

Properties (Definition 2).

$C_G = C_{G-e} - C_{G/e}$;
$C_{G_1 \sqcup G_2} = C_{G_1} \cdot C_{G_2}$, for a disjoint union $G_1 \sqcup G_2$;
$C_\bullet = q$.



Tutte polynomial $T_G(x, y)$.

Definition 1.

$T_G = T_{G-e} + T_{G/e}$ — *if e is neither a bridge nor a loop ;*
$T_G = xT_{G/e}$ — *if e is a bridge ;*
$T_G = yT_{G-e}$ — *if e is a loop ;*
$T_{G_1 \sqcup G_2} = T_{G_1 \cdot G_2} = T_{G_1} \cdot T_{G_2}$ — *for a disjoint union $G_1 \sqcup G_2$ and a one-point join $G_1 \cdot G_2$;*
$T_\bullet = 1$.

Properties.

$T_G(1, 1)$ is the number of spanning trees of G ;
$T_G(2, 1)$ is the number of spanning forests of G ;
$T_G(1, 2)$ is the number of spanning connected subgraphs of G ;
$T_G(2, 2) = 2^{|E(G)|}$ is the number of spanning subgraphs of G .
$C_G(q) = q^{k(G)}(-1)^{r(G)} T_G(1 - q, 0)$.

Definition 2.

Let
- F be a graph;
- $v(F)$ be the number of its vertices;
- $e(F)$ be the number of its edges;
- $k(F)$ be the number of components of F;

- $r(F) := v(F) - k(F)$ be the *rank* of F;
- $n(F) := e(F) - r(F)$ be the *nullity* of F;

$$T_G(x,y) \ := \sum_{F \subseteq E(G)} (x-1)^{r(G)-r(F)}(y-1)^{n(F)}$$

Dichromatic polynomial $Z_G(q,v)$ (Definition 3).

Let $Col(G)$ denote the set of colorings of G with q colors.

$$Z_G(q,v) \ := \sum_{c \in Col(G)} (1+v)^{\#\text{ edges colored not properly by } c}$$

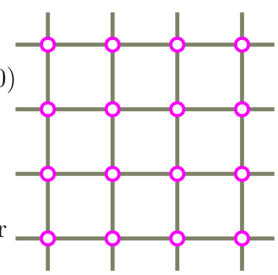
Properties .

$Z_G = Z_{G-e} + vZ_{G/e}$;
$Z_{G_1 \sqcup G_2} = Z_{G_1} \cdot Z_{G_2}$, for a disjoint union $G_1 \sqcup G_2$;
$Z_\bullet = q$;

$$Z_G(q,v) = \sum_{F \subseteq E(G)} q^{k(F)} v^{e(F)} \ ;$$

$C_G(q) = Z_G(q,-1)$;
$Z_G(q,v) = q^{k(G)} v^{r(G)} T_G(1+qv^{-1}, 1+v)$;
$T_G(x,y) = (x-1)^{-k(G)}(y-1)^{-v(G)} Z_G((x-1)(y-1), y-1)$.

Potts model in statistical mechanics (Definition 4).



Potts model (C.Domb 1952); $q=2$ the Ising model (W.Lenz, 1920)

Let G be a graph.

Particles are located at vertices of G. Each particle has a *spin*, which takes q different values . A *state, $\sigma \in \mathcal{S}$,* is an assignment of spins to all vertices of G. Neighboring particles interact with each other only is their spins are the same.

The energy of the interaction along an edge e is $-J_e$ (*coupling constant*). The model is called *ferromagnetic* if $J_e > 0$ and *antiferromagnetic* if $J_e < 0$.

Energy of a state σ (*Hamiltonian*),

$$H(\sigma) = - \sum_{(a,b)=e \in E(G)} J_e \ \delta(\sigma(a), \sigma(b)).$$

Boltzmann weight of σ:

$$e^{-\beta H(\sigma)} = \prod_{(a,b)=e \in E(G)} e^{J_e \beta \delta(\sigma(a),\sigma(b))} = \prod_{(a,b)=e \in E(G)} \Big(1 + (e^{J_e \beta} - 1)\delta(\sigma(a), \sigma(b))\Big),$$

where the *inverse temperature* $\beta = \frac{1}{\kappa \ T}$, T is the temperature, $\kappa = 1.38 \times 10^{-23}$ joules/Kelvin is the *Boltzmann constant*.

The Potts partition function (for $x_e := e^{J_e \beta} - 1$)

$$Z_G(q, x_e) \ := \sum_{\sigma \in \mathcal{S}} e^{-\beta H(\sigma)} = \sum_{\sigma \in \mathcal{S}} \ \prod_{e \in E(G)} (1 + x_e \delta(\sigma(a), \sigma(b)))$$

Properties of the Potts model Probability of a state σ: $P(\sigma) := e^{-\beta H(\sigma)}/Z_G$.
Expected value of a function $f(\sigma)$:

$$\langle f\rangle := \sum_\sigma f(\sigma)P(\sigma) = \sum_\sigma f(\sigma)e^{-\beta H(\sigma)}/Z_G \ .$$

Expected energy: $\langle H\rangle = \sum_\sigma H(\sigma)e^{-\beta H(\sigma)}/Z_G = -\frac{d}{d\beta}\ln Z_G$.

Fortuin—Kasteleyn'1972: $Z_G(q,x_e) = \sum_{F\subseteq E(G)} q^{k(F)}\prod_{e\in F} x_e$,

where $k(F)$ is the number of connected components of the spanning subgraph F.
$Z_G = Z_{G\setminus e} + x_e Z_{G/e}$.

Spanning tree generating function (Definition 5).

For a connected graph G fix an order of its edges: $e_1, e_2, \ldots, e_m$. Let T be a spanning tree.

An edge $e_i \in E(T)$ is called *internally active (live)* if $i < j$ for any edge e_j connecting the two components of $T - e_i$

An edge $e_j \notin E(T)$ is called *externally active (live)* if $j < i$ for any edge e_i in the unique cycle of $T \cup e_j$.

Let $i(T)$ and $j(T)$ be the numbers of internally and externally active edges correspondingly.

$$T_G(x,y) \ := \sum_T \ x^{i(T)}y^{j(T)}$$

Doubly weighted Tutte polynomial .

With each edge e of a graph G we associate two variables (weights) u_e and v_e.

$$T_G(\{u_e,v_e\},x,y) := \sum_{F\subseteq E(G)} \Big(\prod_{e\in F} u_e\Big)\Big(\prod_{e\notin F} v_e\Big)(x-1)^{r(G)-r(F)}(y-1)^{n(F)}$$

Properties.

$T_G = v_e T_{G-e} + u_e T_{G/e}$	if e is neither a bridge nor a loop ;
$T_G = (v_e(x-1)+u_e)T_{G/e}$	if e is a bridge ;
$T_G = (v_e+(y-1)u_e)T_{G-e}$	if e is a loop ;
$T_{G_1\sqcup G_2} = T_{G_1\cdot G_2} = T_{G_1}\cdot T_{G_2}$	for a disjoint union $G_1\sqcup G_2$ and a one-point join $G_1\cdot G_2$;
$T_\bullet = 1$.	

Tutte polynomial of signed graphs .

Signed graph is a a graphs with signs ±1 assigned to the edges of the graph.
We define the Tutte polynomial of a signed graph by substituting the following weights to the doubly weighted Tutte polynomial.

$$+\text{-edge: } u_e := 1,\quad v_e := 1; \qquad -\text{-edge: } u_e := \sqrt{\frac{x-1}{y-1}},\qquad v_e := \sqrt{\frac{y-1}{x-1}}.$$

With this substitution the Tutte polynomial for signed graphs becomes

$$T_G(x,y) = \sum_{F\subseteq E(G)} (x-1)^{r(G)-r(F)+s(F)}(y-1)^{n(F)-s(F)} \ ,$$

for $s(F) := \dfrac{e_-(F) - e_-(E(G)\setminus F)}{2}$, where $e_-(S)$ stands for the number of negative edges of S.

Chromatic polynomial of signed graphs.

There are two chromatic polynomials of signed graphs.

A q-*coloring* of a signed G is a map $c: V(G) \to \{-q, -q+1, \ldots, -1, 0, 1, \ldots, q-1, q\}$. A q-coloring c is *proper* if for any edge e with the sign ε_e: $c(v_1) \neq \varepsilon c(v_2)$, where v_1 and v_2 are the endpoints of e.

Definition .
$C_G(2q+1) :=$ # of proper q-colorings of G.
$C_G^{\neq 0}(2q) :=$ # of proper q-colorings of G which take nonzero values.

Properties.

- $C_G(\lambda)$ is a polynomial function of $\lambda = 2q+1 > 0$;
- $C_G^{\neq 0}(\lambda)$ is a polynomial function of $\lambda = 2q > 0$;
- $C_G(\lambda) = C_{G-e}(\lambda) - C_{G/e}(\lambda)$;
- $C_G^{\neq 0}(\lambda) = C_{G-e}^{\neq 0}(\lambda) - C_{G/e}^{\neq 0}(\lambda)$;
- $C_{G_1\sqcup G_2} = C_{G_1}\cdot C_{G_2}$ and $C_{G_1\sqcup G_2}^{\neq 0} = C_{G_1}^{\neq 0}\cdot C_{G_2}^{\neq 0}$ for a disjoint union $G_1 \sqcup G_2$;
- $C_\emptyset = 1$.

Problem.
Express the chromatic polynomials of signed graphs it terms of the signed Tutte polynomial.