Dual representable matroids

Let $E = \{v_1, \ldots, v_n\}$ be a collection of vectors in a vector space U and M be a matroid of their linear dependences. Consider an n-dimensional vector space V with a basis $e_1, \ldots, e_n$ and a linear map $f : V \to U$ sending e_k to v_k. Denote the kernel of this map by W. It is a subspace of V and there is a natural inclusion map $i : W \hookrightarrow V$. There is the dual map $W^* \xleftarrow{i^*} V^*$ of dual vector spaces. The space V^* has a natural dual basis $e_1^*, \ldots, e_n^*$. Their images $i^*(e_1^*), \ldots, i^*(e_n^*)$ is a collection of vectors in the space W^*. These vectors with the structure of linear dependences between them form the dual matroid M^*.

Δ-matroids [Bouchet]

Matroids	Δ-matroids
A matroid is a pair $M = (E, \mathcal{B})$ consisting of a finite set E and a nonempty collection $\mathcal{B}$ of its subsets, called *bases*, satisfying the axioms: **(B1)** *No proper subset of a base is a base.* **(B2)** =(**Exchange axion**) *If B_1 and B_2 are bases and $b_1 \in B_1 - B_2$, then there is an element $b_2 \in B_2 - B_1$ such that $(B_1 - b_1) \cup b_2$ is a base.*	*A Δ-matroid* is a pair $M = (E, \mathcal{F})$ consisting of a finite set E and a nonempty collection $\mathcal{F}$ of its subsets, called *feasible sets*, satisfying the **Symmetric Exchange axion** *If F_1 and F_2 are two feasible sets and $f_1 \in F_1 \Delta F_2$, then there is an element $f_2 \in F_1 \Delta F_2$ such that $F_1 \Delta \{f_1, f_2\}$ is a feasible set.*

Ribbon graphs (graphs on surfaces)

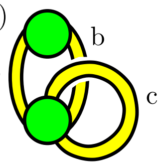
Definition. A *ribbon graph* G is a surface (possibly non-orientable) with boundary, represented as the union of two sets of closed topological discs called *vertices* $V(G)$ and *edges* $E(G)$, satisfying the following conditions:

- these vertices and edges intersect by disjoint line segments;
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- every edge contains exactly two such line segments.

Definition. A *quasi-tree* is ribbon graph G with a single boundary component, $bc(G) = 1$.

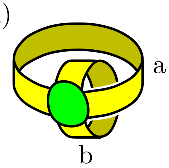
Examples.

(i)



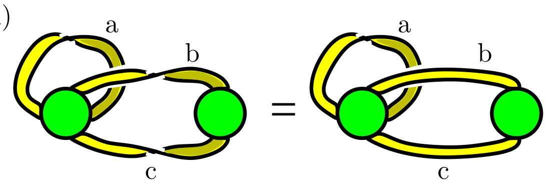
Spanning quasi-trees: $\{a\}, \{b\}, \{a, b, c\}$

(ii)



Spanning quasi-trees: $\emptyset, \{a, b\}$

(iii)



Spanning quasi-trees: $\{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}$

Theorem. *Let $G = (V, E)$ be a ribbon graph. Then $D(G) := (E, \{\text{spanning quasi-trees}\})$ is a Δ-matroid.*

Minors in Δ-matroids

Let $D = (E, \mathcal{F})$ be a Δ-matroid and $e \in E$.
e is a *loop* iff $\forall F \in \mathcal{F}, e \notin F$. e is a *coloop* iff $\forall F \in \mathcal{F}, e \in F$.
If e is not a loop, $D/e := (E \setminus \{e\}, \{F \setminus \{e\} | F \in \mathcal{F}, e \in F\})$.
If e is not a coloop, $D \setminus e := (E \setminus \{e\}, \{F | F \in \mathcal{F}, F \subset E \setminus \{e\}\})$.

Twists of Δ-matroids. Let $D = (E, \mathcal{F})$ be a Δ-matroids and $A \subset E$.

$$D * A := (E, \{F \Delta A | F \in \mathcal{F}\}).$$

Dual Δ-matroid: $D^* := D * E$.

Matroids associated with a Δ-matroid

Let $D = (E, \mathcal{F})$ be a Δ-matroid.
$D_{min} := (E, \mathcal{F}_{min})$, where $\mathcal{F}_{min} := \{F \in \mathcal{F} | F$ is of minimal possible cardinality$\}$.
$D_{max} := (E, \mathcal{F}_{max})$, where $\mathcal{F}_{max} := \{F \in \mathcal{F} | F$ is of maximal possible cardinality$\}$.

Facts.

- D_{min} and D_{max} are usual matroids. *Width* $w(D) := r(D_{max}) - r(D_{min})$.
- $(D(G))_{min} = \mathcal{C}(G)$. $(D(G))_{max} = (\mathcal{C}(G^*))^*$.
- $D(G) = \mathcal{C}(G)$ iff G is a planar ribbon graph.

Matroid perspectives (M. Las Vergnas [LV])

Definition. Let M and M' be two matroid structures on the same ground set E. They form a *matroid perspective* $M \to M'$ if any circuit of M is a union of circuits of M'. Equivalently,

$$r_M(X) - r_M(Y) \geqslant r_{M'}(X) - r_{M'}(Y) \qquad \text{for all} \quad Y \subseteq X \subseteq E.$$

Example.
For graphs G and G^* dually embedded in a surface, then the bond matroid of G^* and the circuit matroid of G form a matroid perspective, $\mathcal{B}(G^*) \to \mathcal{C}(G)$.

Lemma. *For any Δ-matroid D, $D_{max} \to D_{min}$ is a matroid perspective.*

Tutte like polynomials

The Las Vergnas polynomial ([LV]) of a matroid perspective $M \to M'$.

$$T_{M \to M'}(x, y, z) := \sum_{X \subseteq M} (x-1)^{r_{M'}(E) - r_{M'}(X)} (y-1)^{n_M(X)} z^{(r_M(E) - r_M(X)) - (r_{M'}(E) - r_{M'}(X))}$$

Properties. $T_M(x, y) = T_{M \to M}(x, y, z)$; $T_{M'}(x, y) = (y-1)^{r_M(E) - r_{M'}(E)} T_{M \to M'}(x, y, \frac{1}{y-1})$.

The Bollobás-Riordan polynomial ([BR]) of a ribbon graph G.

$$R_G(X, Y, Z) := \sum_{F \subseteq G} X^{r(G) - r(F)} Y^{n(F)} Z^{k(F) - \mathrm{bc}(F) + n(F)}$$

References

[BR] B. Bollobás and O. Riordan, *A polynomial of graphs on surfaces*, Math. Ann. **323** (2002) 81–96.
[Bouchet] A. Bouchet, *Greedy algorithm and symmetric matroids*, Math. Program. **38** (1987) 147–159.
[LV] M. Las Vergnas, *On the Tutte polynomial of a morphism of matroids*, Annals of Discrete Mathematics **8** (1980) 7–20.