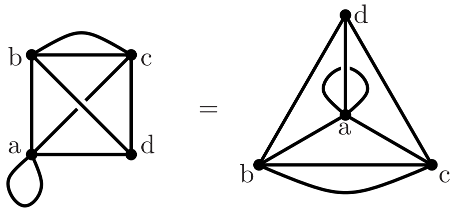
## Graphs

**Definition.** A *graph* $G$ is a finite set of *vertices* $V(G)$ and a finite set $E(G)$ of unordered pairs $(x, y)$ of vertices $x, y \in V(G)$ called *edges*.

A graph may have *loops* $(x, x)$ and *multiple edges* when a pair $(x, y)$ appears in $E(G)$ several times. Pictorially we represent the vertices by points and edges by lines connecting the corresponding points. Topologically a graph is a 1-dimensional *cell complex* with $V(G)$ as the set of 0-cells and $E(G)$ as the set of 1-cells. Here are two pictures representing the same graph.




$$V(G) = \{a, b, c, d\}$$

$$E(G) = \{(a, a), (a, b), (a, c), (a, d),\\ (b, c), (b, c), (b, d), (c, d)\}$$

## Tutte polynomial

***Chromatic polynomial*** $C_G(q)$.

A *coloring* of $G$ with $q$ colors is a map $c : V(G) \to \{1, \ldots, q\}$. A coloring $c$ is *proper* if for any edge $e$: $c(v_1) \neq c(v_2)$, where $v_1$ and $v_2$ are the endpoints of $e$.

**Definition 1.** *$C_G(q) :=$ # of proper colorings of $G$ in $q$ colors.*

**Properties (Definition 2).**

$C_G = C_{G-e} - C_{G/e}$ ;

$C_{G_1 \sqcup G_2} = C_{G_1} \cdot C_{G_2}$, for a disjoint union $G_1 \sqcup G_2$ ;

$C_\bullet = q$ .

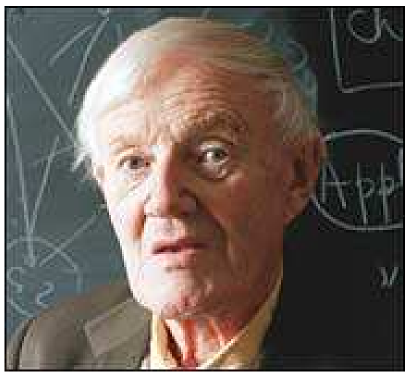
***Flow polynomial*** $Q_G(q)$.

A *q-flow* on $G$ is an assignment of a value $0, 1, \ldots, q - 1$ to every edge of $G$ with arbitrarily chosen orientation of its edges in such a way that the total flow entering and leaving each vertex is congruent modulo $q$.

**Definition 1.** *$Q_G(q) :=$ # of nowhere-zero q-flows on $G$.*

**Properties (Definition 2).**

| | |
|---|---|
| $Q_G(q) = 0$ | if $G$ has a bridge ; |
| $Q_G(q) = (q - 1)Q_{G-e}(q)$ | if $e$ is a loop ; |
| $Q_G(q) = -Q_{G-e}(q) + Q_{G/e}$ | if $e$ is neither a bridge nor a loop ; |
| $Q_G(q) = C_{G^*}(q)/q$ | for dual planar graphs $G$ and $G^*$ . |



***Tutte polynomial*** $T_G(x, y)$.

**Definition 1.**

| | |
|---|---|
| $T_G = T_{G-e} + T_{G/e}$ | *if $e$ is neither a bridge nor a loop ;* |
| $T_G = xT_{G/e}$ | *if $e$ is a bridge ;* |
| $T_G = yT_{G-e}$ | *if $e$ is a loop ;* |
| $T_{G_1 \sqcup G_2} = T_{G_1 \cdot G_2} = T_{G_1} \cdot T_{G_2}$ | *for a disjoint union $G_1 \sqcup G_2$ and a one-point join $G_1 \cdot G_2$ ;* |

$T_\bullet = 1$ .

**Properties.**

$T_G(1,1)$ is the number of spanning trees of $G$ ;
$T_G(2,1)$ is the number of spanning forests of $G$ ;
$T_G(1,2)$ is the number of spanning connected subgraphs of $G$ ;
$T_G(2,2) = 2^{|E(G)|}$ is the number of spanning subgraphs of $G$ ;
$C_G(q) = q^{k(G)}(-1)^{r(G)} T_G(1-q, 0)$ ;
$Q_G(q) = (-1)^{n(G)} T_G(0, 1-q)$ .

**Definition 2.**

Let
- $F$ be a graph;
- $v(F)$ be the number of its vertices;
- $e(F)$ be the number of its edges;
- $k(F)$ be the number of connected components of $F$;
- $r(F) := v(F) - k(F)$ be the *rank* of $F$;
- $n(F) := e(F) - r(F)$ be the *nullity* of $F$;

$$\boxed{T_G(x,y) \ := \sum_{F \subseteq E(G)} (x-1)^{r(G)-r(F)} (y-1)^{n(F)}}$$

**Dichromatic polynomial $Z_G(q,v)$ (Definition 3).**

Let $Col(G)$ denote the set of colorings of $G$ with $q$ colors.

$$\boxed{Z_G(q,v) \ := \sum_{c \in Col(G)} (1+v)^{\#\text{ edges colored not properly by } c}}$$

**Properties .**

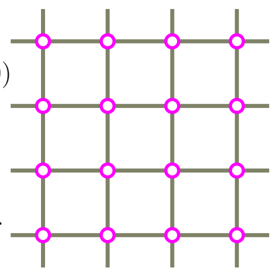
$Z_G = Z_{G-e} + v Z_{G/e}$ ;
$Z_{G_1 \sqcup G_2} = Z_{G_1} \cdot Z_{G_2}$ , for a disjoint union $G_1 \sqcup G_2$ ;
$Z_\bullet = q$ ;

$$Z_G(q,v) = \sum_{F \subseteq E(G)} q^{k(F)} v^{e(F)} \ ;$$

$$C_G(q) == Z_G(q,-1) \ ;$$

$$Z_G(q,v) = q^{k(G)} v^{r(G)} T_G(1 + qv^{-1}, 1+v) \ ;$$
$$T_G(x,y) = (x-1)^{-k(G)} (y-1)^{-v(G)} Z_G((x-1)(y-1), y-1) \ .$$

**Potts model in statistical mechanics (Definition 4).**



Potts model (C.Domb 1952); $q = 2$ the Ising model (W.Lenz, 1920)

Let $G$ be a graph.

Particles are located at vertices of $G$. Each particle has a *spin*, which takes $q$ different values . A *state, $\sigma \in \mathcal{S}$,* is an assignment of spins to all vertices of $G$. Neighboring particles interact with each other only is their spins are the same.
The energy of the interaction along an edge $e$ is $-J_e$ (*coupling constant*). The model is called *ferromagnetic* if $J_e > 0$ and *antiferromagnetic* if $J_e < 0$.

Energy of a state $\sigma$ (*Hamiltonian*),

$$H(\sigma) = -\sum_{(a,b)=e\in E(G)} J_e\ \delta(\sigma(a),\sigma(b)).$$

*Boltzmann weight* of $\sigma$:

$$e^{-\beta H(\sigma)} = \prod_{(a,b)=e\in E(G)} e^{J_e\beta\delta(\sigma(a),\sigma(b))} = \prod_{(a,b)=e\in E(G)} \left(1+(e^{J_e\beta}-1)\delta(\sigma(a),\sigma(b))\right),$$

where the *inverse temperature* $\beta = \frac{1}{\kappa\ T}$, $T$ is the temperature, $\kappa = 1.38\times 10^{-23}$ joules/Kelvin is the *Boltzmann constant*.

The Potts partition function (for $x_e := e^{J_e\beta}-1$)

$$\boxed{Z_G(q,x_e)\ := \sum_{\sigma\in\mathcal{S}} e^{-\beta H(\sigma)} = \sum_{\sigma\in\mathcal{S}}\ \prod_{e\in E(G)} (1+x_e\delta(\sigma(a),\sigma(b)))}$$

**Properties of the Potts model** Probability of a state $\sigma$: $P(\sigma) := e^{-\beta H(\sigma)}/Z_G$ .
Expected value of a function $f(\sigma)$:

$$\langle f\rangle := \sum_\sigma f(\sigma)P(\sigma) = \sum_\sigma f(\sigma)e^{-\beta H(\sigma)}/Z_G\ .$$

Expected energy: $\langle H\rangle = \sum_\sigma H(\sigma)e^{-\beta H(\sigma)}/Z_G = -\frac{d}{d\beta}\ln Z_G$ .

Fortuin—Kasteleyn'1972: $Z_G(q,x_e) = \sum_{F\subseteq E(G)} q^{k(F)}\prod_{e\in F} x_e$ ,

where $k(F)$ is the number of connected components of the spanning subgraph $F$.
$Z_G = Z_{G\setminus e} + x_e Z_{G/e}$ .

**Spanning tree generating function (Definition 5).**

For a connected graph $G$ fix an order of its edges: $e_1, e_2, \ldots, e_m$. Let $T$ be a spanning tree.

An edge $e_i\in E(T)$ is called *internally active (live)* if $i<j$ for any edge $e_j$ connecting the two components of $T-e_i$

An edge $e_j\notin E(T)$ is called *externally active (live)* if $j<i$ for any edge $e_i$ in the unique cycle of $T\cup e_j$.

Let $i(T)$ and $j(T)$ be the numbers of internally and externally active edges correspondingly.

$$\boxed{T_G(x,y)\ := \sum_T x^{i(T)}y^{j(T)}}$$

***Doubly weighted Tutte polynomial*** .

With each edge $e$ of a graph $G$ we associate two variables (weights) $u_e$ and $v_e$.

$$\boxed{T_G(\{u_e,v_e\},x,y) := \sum_{F\subseteq E(G)} (\prod_{e\in F} u_e)\ (\prod_{e\notin F} v_e)(x-1)^{r(G)-r(F)}(y-1)^{n(F)}}$$

**Properties.**

$$\begin{array}{ll} T_G = v_e T_{G-e} + u_e T_{G/e} & \text{if } e \text{ is neither a bridge nor a loop ;} \\ T_G = (v_e(x-1)+u_e) T_{G/e} & \text{if } e \text{ is a bridge ;} \\ T_G = (v_e + (y-1)u_e) T_{G-e} & \text{if } e \text{ is a loop ;} \\ T_{G_1 \sqcup G_2} = T_{G_1 \cdot G_2} = T_{G_1} \cdot T_{G_2} & \text{for a disjoint union } G_1 \sqcup G_2 \\ & \text{and a one-point join } G_1 \cdot G_2 \text{ ;} \\ T_\bullet = 1 \ . & \end{array}$$

***Tutte polynomial of signed graphs*** .

Signed graph is a a graphs with signs $\pm 1$ assigned to the edges of the graph.

We define the Tutte polynomial of a signed graph by substituting the following weights to the doubly weighted Tutte polynomial.

$$+\text{-edge: } u_e := 1, \quad v_e := 1; \qquad -\text{-edge: } u_e := \sqrt{\frac{x-1}{y-1}}, \qquad v_e := \sqrt{\frac{y-1}{x-1}}.$$

With this substitution the Tutte polynomial for signed graphs becomes

$$T_G(x,y) = \sum_{F \subseteq E(G)} (x-1)^{r(G)-r(F)+s(F)} (y-1)^{n(F)-s(F)} \ ,$$

for $s(F) := \dfrac{e_-(F) - e_-(E(G) \setminus F)}{2}$ , where $e_-(S)$ stands for the number of negative edges of $S$.

***Chromatic polynomial of signed graphs***.

There are two chromatic polynomials of signed graphs.

A $q$-*coloring* of a signed $G$ is a map $c : V(G) \to \{-q, -q+1, \ldots, -1, 0, 1, \ldots, q-1, q\}$. A $q$-coloring $c$ is *proper* if for any edge $e$ with the sign $\varepsilon_e$: $c(v_1) \neq \varepsilon c(v_2)$, where $v_1$ and $v_2$ are the endpoints of $e$.

**Definition .**
*$C_G(2q+1) :=$ # of proper $q$-colorings of $G$.*
*$C_G^{\neq 0}(2q) :=$ # of proper $q$-colorings of $G$ which take nonzero values.*

**Properties.**

- $C_G(\lambda)$ is a polynomial function of $\lambda = 2q+1 > 0$ ;
- $C_G^{\neq 0}(\lambda)$ is a polynomial function of $\lambda = 2q > 0$ ;
- $C_G(\lambda) = C_{G-e}(\lambda) - C_{G/e}(\lambda)$ ;
- $C_G^{\neq 0}(\lambda) = C_{G-e}^{\neq 0}(\lambda) - C_{G/e}^{\neq 0}(\lambda)$ ;
- $C_{G_1 \sqcup G_2} = C_{G_1} \cdot C_{G_2}$ and $C_{G_1 \sqcup G_2}^{\neq 0} = C_{G_1}^{\neq 0} \cdot C_{G_2}^{\neq 0}$ for a disjoint union $G_1 \sqcup G_2$ ;
- $C_\emptyset = 1$ .

**Problems.**

**1.** Express the chromatic polynomials of signed graphs it terms of the signed Tutte polynomial.

**2.** Make a definition of the flow polynomial for signed graphs and relate it to the signed Tutte polynomial.