Knots and Graphs

Working Group [Summer 2017]

MATH 4193, class number 17381

Instructor: ***Sergei Chmutov***

RESEARCH PROJECTS

Project 1. ***Knots and links in spatial graphs.*** (Maxwell Budd, Caleb Dilsavor, Ningyi Liu)

This is a continuation of the last year project. It is based on a famous paper [CG] by John Conway and Cameron Gordon showing that every embedding of the complete graph K_6 in $\mathbb{R}^3$ contains at least one pair of linked triangles and every embedding of K_7 contains a non-trivial knot. An excellent exposition of this results see in [Ad, Ch.8].

One of the result of the last year was that the graph K_8 can be embedded in $\mathbb{R}^3$ in such a way that there are two cycles with a linking number one. On the other hand, it is known [FNP, Flap] that any embedding of K_{10} contains two cycles with the linking number at least 2. The situation with K_9 is unknown. We are going to try to fix it. Also we will try to improve the estimate of [ST] for the number of vertices of a complete graph whose arbitrary embedding has to contain two cycles with the linking number at least n?

Project 2. ***Skein invariants.*** (Will Hoffer, Adu Vengal, Andersen Weaver)

A *skein relation* is an equation for values a link invariant on locally different links. Polynomial knot (link) invariants usually satisfy some skein relation. The famous Jones polynomial can be defined by means of skein relation due L.Kauffman, [Ad, Ch.6] and [Ka1].

Recently there was an attempt to generalize Kauffman's skein relations [Yang]. The goal of the project is to analyze these attempts and try to produce new invariants of classical and virtual links in this or similar way.

Project 3. ***Tropical geometry.*** (Dan Brogan, Nik Henderson, Vilas Winstein)

Classical algebraic geometry studies the solution sets of systems of polynomial equations in several variables. It depends on the field over which we consider our polynomial. The common choices are a real algebraic geometry or a complex algebraic geometry. Tropical geometry comes from replacing the ground field by tropical semi-field $\mathbb{T}$ where the addition is defined as maxumum: "$x + y$":$= \max(x, y)$, and multiplication is defined by usual sum: "$x \times y$":$= x + y$. Thus the tropical single variable polynomial becomes a piecewise linear function: $P(x) = \text{“} \sum_{n=0}^{d} a_n x^n \text{”} = \max_{n=0}^{d}(a_n + nx)$. For a good introduction to the tropical geometry see [BrSh, Mikh].

This project is a continuation of the Spring semester work. We are working with a concept of Weierstrass points on metric graphs. In particular we are investigating whether the set of all Weierstrass points could be countable. The main references here are [Amin, HMY]

Project 4. ***Virtual links.*** (Michael Crawshaw, Michael Heinz, Aidan Howells)

This is related to the the project #2. In 2009 L. Kauffman [Ka2] suggested a generalization of the Jones polynomial for virtual links with values in the linear combinations of certain graphs with polynomial coefficients. It happen that Kauffman's theorem contains a gap discovered by Benjamin OConnor [Ch]. L. Kauffman [Ka3] fixed the gap considering instead of graphs the classes of equivalences of graphs. However he didn't have any algorithm deciding whether two graphs are equivalent or not. We will try to complete this research and look for a possible algorithm.

References

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