Symmetric functions.

Definition. A *symmetric function* on n (may be infinitely many) variables $x_1, \dots, x_n$ is a function that is unchanged by any permutation of its variables. If π is a permutation of $\{1, 2, \dots, n\}$, then for a symmetric function $f(x_1, \dots, x_n)$ we have $f(x_1, \dots, x_n) = f(x_{\pi(1)}, \dots, x_{\pi(n)})$ So it is invariant under the action of the *symmetric group* S_n acting on the indices of variables.

The symmetric functions form a *vector space* $\mathbb{R}[x_1, \dots, x_n]^{S_n}$. Moreover, they form an *algebra* (a vector space with an operation of multiplication of vectors).

Elementary symmetric functions.

$$\begin{aligned}
e_0 &:= 1;\\
e_1 &:= x_1 + x_2 + \cdots + x_n;\\
e_2 &:= (x_1x_2 + x_1x_3 + \cdots + x_1x_n) + (x_2x_3 + \cdots + x_2x_n) + \cdots + x_{n-1}x_n;\\
e_3 &:= x_1x_2x_3 + \dots\cdots + x_{n-2}x_{n-1}x_n;\\
&\vdots\\
e_n &:= x_1x_2\dots x_{n-1}x_n.
\end{aligned}$$

In general $$\boxed{e_k := \sum_{1 \le j_1 < j_2 \dots j_k \le n} x_{j_1} \dots x_{j_k}}.$$

Vieta's formulas: the elementary symmetric polynomials in $x_1, \dots, x_n$ are coefficients (up to the sign) of a polynomial $p(x)$ with roots $x_1, \dots, x_n$:

$$p(x) = x^n - e_1x^{n-1} + e_2x^{n-2} - \cdots \pm e_{n-1}x \mp e_n = (x - x_1)(x - x_2)\dots(x - x_n).$$

Fundamental theorem of symmetric polynomials.
Any symmetric polynomial $p(x_1, \dots, x_n)$ can be uniquely(!!!) represented as a polynomial in the elementary symmetric functions: $p(x_1, \dots, x_n) = q(e_1, \dots, e_n)$ for an appropriate polynomial $q(y_1, \dots, y_n)$.

In other words, the algebra of symmetric polynomials $\mathbb{R}[x_1, \dots, x_n]^{S_n}$ is isomorphic to the algebra of polynomials $\mathbb{R}[y_1, \dots, y_n]$. Thus, the elementary symmetric polynomials $e_1, \dots, e_n$ constitute the *generating set* for the algebra $\mathbb{R}[x_1, \dots, x_n]^{S_n}$. As a vector space, it has a basis (linear, or additive) consisting of all monomials in $e_1, \dots, e_n$.

Symmetric power functions

$$\boxed{p_k(x_1, \dots, x_n) := x_1^k + x_2^k + \cdots + x_n^k}$$

also form a generating set for $\mathbb{R}[x_1, \dots, x_n]^{S_n}$.

Newton's identities.

$$ke_k = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i} p_i.$$