# A Categorification of Biquandle Brackets

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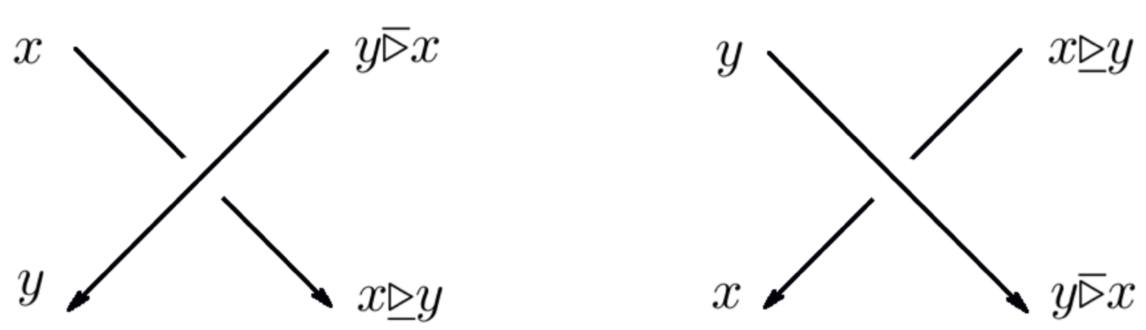
## Biquandles

A *biquandle* is a set $X$ with two binary operations $\trianglerighteq, \overline{\triangleright}$ such that for every $x, y, z \in X$,

1. $x \trianglerighteq x = x \,\overline{\triangleright}\, x$
2. The maps $\alpha_y(x) = x \,\overline{\triangleright}\, y$, $\beta_y(x) = x \trianglerighteq y$, and $S(x,y) = (y \,\overline{\triangleright}\, x, x \trianglerighteq y)$ are invertible.
3. The following exchange laws are satisfied:

$$(x \trianglerighteq y) \trianglerighteq (z \trianglerighteq y) = (x \trianglerighteq z) \trianglerighteq (y \,\overline{\triangleright}\, z)$$
$$(x \trianglerighteq y) \,\overline{\triangleright}\, (z \trianglerighteq y) = (x \,\overline{\triangleright}\, z) \trianglerighteq (y \,\overline{\triangleright}\, z)$$
$$(x \,\overline{\triangleright}\, y) \,\overline{\triangleright}\, (z \,\overline{\triangleright}\, y) = (x \,\overline{\triangleright}\, z) \,\overline{\triangleright}\, (y \trianglerighteq z).$$

The biquandle axioms mirror the three Reidemeister moves for link diagrams. For example, if the strands of a link are colored by elements of a biquandle so that the following relations hold at each crossing:




then the biquandle axioms are exactly what is required to allow such a coloring to be invariant under the Reidemeister moves.

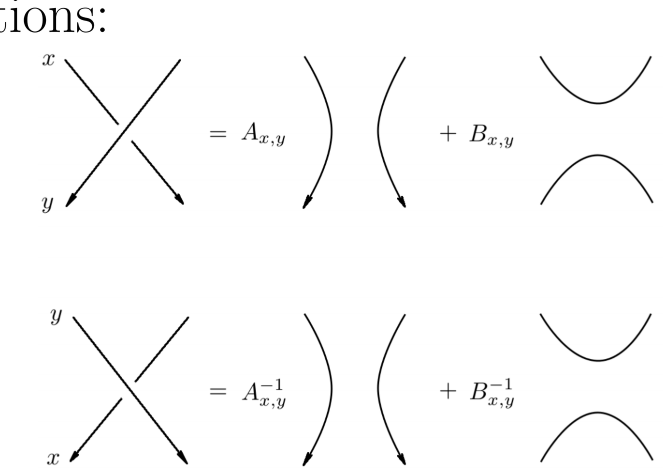
Examples of biquandles include

- *The trivial biquandle*: $X = \{x\}$ is a singleton, with operations $x \,\overline{\triangleright}\, x = x \trianglerighteq x = x$.
- *Constant action biquandles*: $X$ is any set, $\sigma : X \to X$ is any bijection. The operations are $x \trianglerighteq y = x \,\overline{\triangleright}\, y = \sigma(x)$ for all $x, y \in X$.
- *Alexander biquandles*: $X$ is any module over $\mathbb{Z}[t^{\pm 1}, r^{\pm 1}]$. Then $x \trianglerighteq y = tx + (r - t)\,y$ and $x \,\overline{\triangleright}\, y = ry$ define a biquandle.

Given any biquandle $X$, the number of $X$-colorings of a link diagram is an invariant and is called the *biquandle counting invariant*. If $L$ is a link, the $X$-counting invariant of $L$ is $\Phi_X^{\mathbb{Z}}(L)$.

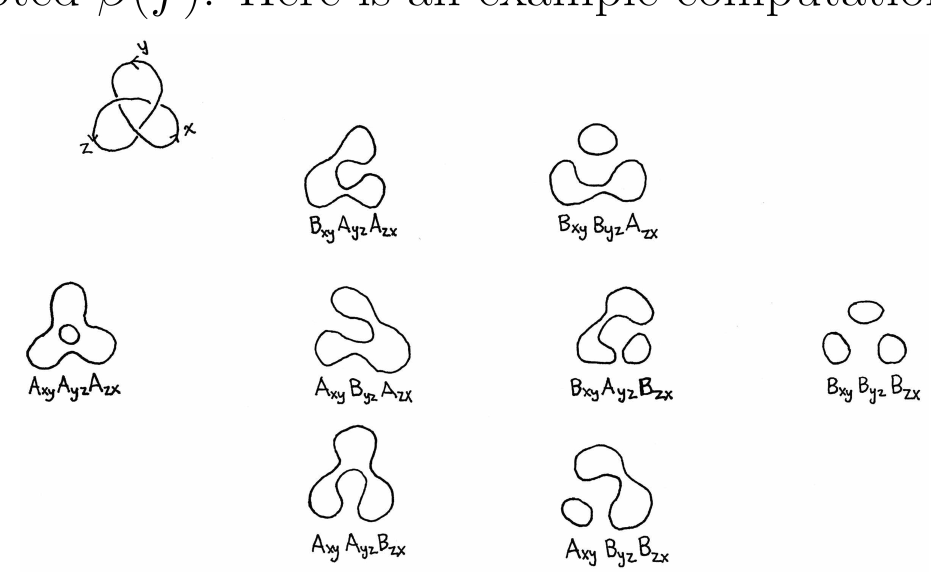
## Biquandle Brackets

Let $X$ be a biquandle and $R$ a commutative unital ring. We would like to choose elements $A_{x,y}, B_{x,y} \in R^\times$ (for each $x, y \in X$), $w \in R^\times$, and $\delta \in R$ such that the element of $R$ determined by the following skein relations:




with $\delta$ the value of a circle and $w$ the value of a positive kink, is an invariant of $X$-colored links. Such a collection of elements of $R$ is called a *biquandle bracket* and must satisfy three axioms found in [1].

It turns out that that $w$ and $\delta$ are determined by $A$ and $B$. So a biquandle bracket is denoted $\beta = (A, B)$ and it's value on a coloring $f$ of a link is denoted $\beta(f)$. Here is an example computation:




$$\beta(f) = w^{-3}\left( \delta^2 A_{x,y}A_{y,z}A_{z,x} \begin{matrix} +\delta B_{x,y}A_{y,z}A_{z,x} \\ +\delta A_{x,y}B_{y,z}A_{z,x} \\ +\delta A_{x,y}A_{y,z}B_{z,x} \end{matrix} \begin{matrix} +\delta^2 B_{x,y}B_{y,z}A_{z,x} \\ +\delta^2 B_{x,y}A_{y,z}B_{z,x} \\ +\delta^2 A_{x,y}B_{y,z}B_{z,x} \end{matrix} +\delta^3 B_{x,y}B_{y,z}B_{z,x} \right)$$

The following multiset-valued function on links is is link invariant which enhances $\Phi_X^{\mathbb{Z}}(L)$:

$$\Phi_X^\beta(L) = \{\beta(f) : f \text{ is an } X\text{-coloring of } L\}.$$

When $X = \{x\}$ is the trivial biquandle, we can let $R = \mathbb{Z}[q^{\pm 1}]$, and let $A_{x,x} = q$ and $B_{x,x} = q^{-1}$ to obtain the Jones polynomial as the single element of $\Phi_X^\beta(L)$. Additionally, using a computer, one can find many other biquandles and many many other biquandle brackets.

## Our Construction

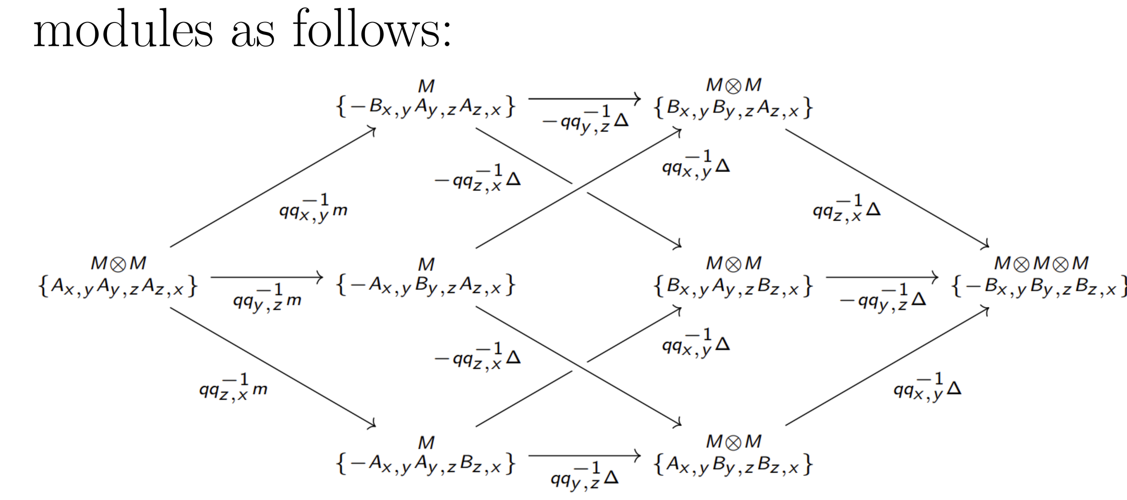
Let $\beta = (A, B)$ be an $X$-bracket with values in $R$.

1. Let $q_{x,y} = -\frac{B_{x,y}}{A_{x,y}}$ for all $x, y \in X$.
2. Let $x_0 \in X$, and let $q = q_{x_0,x_0}$.
3. Let $G$ be the group $\langle q q_{x,y}^{-1} : x, y \in X\rangle \leq R^\times$.
4. Let $S$ be the $R^\times$-graded group algebra $\mathbb{Z}[G]$, with the $R^\times$-grading given by $\deg(g) = g$ for all $g \in G$.
5. Let $M$ be the $R^\times$ graded $S$-module $S[t]/(t^2)$ with the additional grading given by $\deg(1) = q$ and $\deg(t) = q^{-1}$. $M$ is a Frobenius algebra with the following multiplication and comultiplication:

$$m : M \otimes M \to M$$
$$m : 1 \otimes 1 \mapsto 1, \quad 1 \otimes t \mapsto t,$$
$$t \otimes 1 \mapsto t, \quad t \otimes t \mapsto 0$$

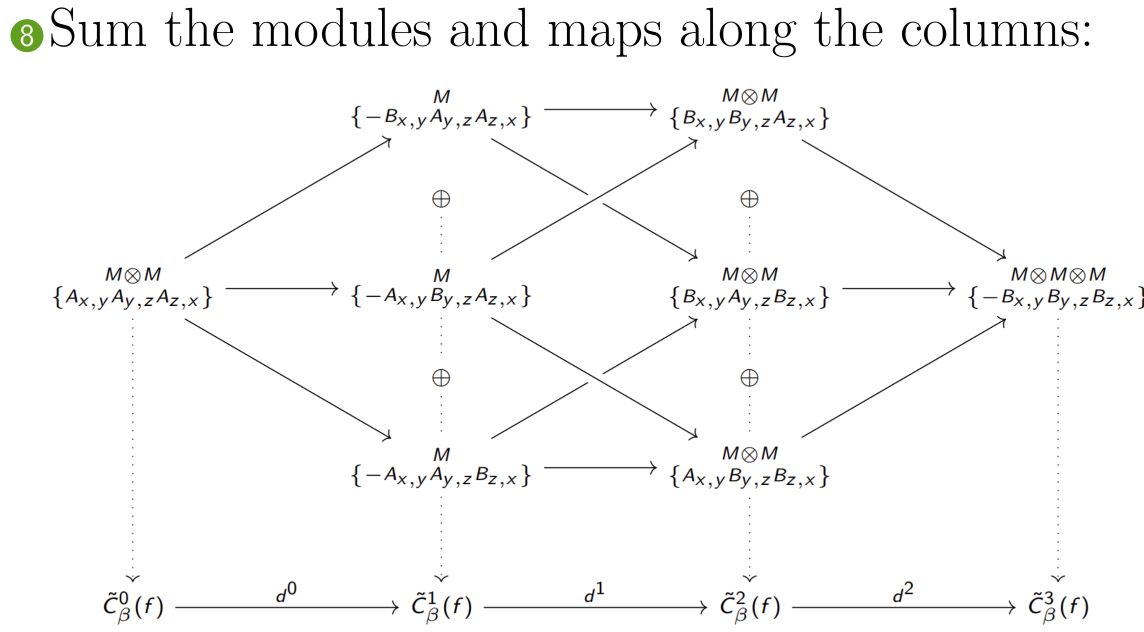
$$\Delta : M \to M \otimes M$$
$$\Delta : 1 \mapsto 1 \otimes t + t \otimes 1, \quad t \mapsto t \otimes t$$

6. Let $L$ be a link and let $f$ be an $X$-coloring of $L$. Perform splittings as in the figure to the left.
7. Replace circles with copies of $M$ and tensor adjacent copies together. Add maps between the modules as follows:



This gives a cube with anti-commutative faces.

8. Sum the modules and maps along the columns:



to obtain $\tilde{C}_\beta(f)$, an $R^\times$ graded cochain complex.

## Our Construction, Continued

9. Let $C_\beta(f)$ be the shifted cochain complex:

$$C_\beta(f) = \tilde{C}_\beta(f)[n_-]\{(-1)^{n_-} w^{-n_+} w^{n_-}\},$$

where $n_\pm$ is the number of positive or negative crossings in the link.

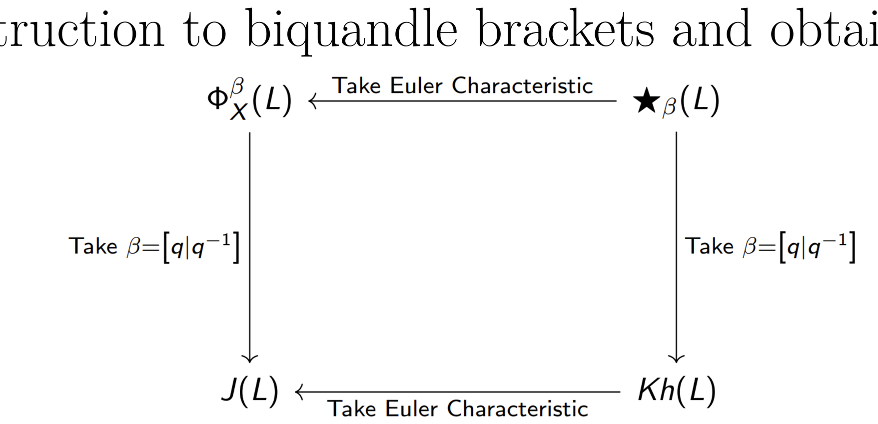
10. Finally, take the cohomology of $C_\beta(f)$ to obtain a complex $\mathcal{H}_\beta(f)$. The multiset

$$Bh_\beta(L) = \{\mathcal{H}_\beta(f) : f \text{ is an } X\text{-coloring of } L\}$$

is an invariant of links.

## Results

The biquandle bracket generalizes the Jones polynomial, which Khovanov's homology construction [2] categorifies. Our goal was to generalize Khovanov's construction to biquandle brackets and obtain ★:




The Euler characteristic of $Bh_\beta(L)$ is actually $\mathrm{rdim}(S) \cdot \Phi_X^\beta(L)$, where $\mathrm{rdim}(S)$ is the $R^\times$-graded dimension of $S$ which cannot always be factored out to yield $\Phi_X^\beta(L)$. In fact, $Bh_\beta(L)$ is actually isomorphic to a quotient of $Kh(L)$ shifted by

$$\mathrm{rdim}(S)\left(\prod_{\tau^+} A_{x,y}A^{-1}_{x_0,x_0}\right)\left(\prod_{\tau^-} B^{-1}_{x,y}B_{x_0,x_0}\right).$$

This shift is the value of a *biquandle 2-cocycle invariant* which takes values in $R^\times/G$. Further work will explore how this new invariant compares to the related biquandle bracket.

## References

[1] Sam Nelson, Michael E Orrison, and Veronica Rivera. Quantum enhancements and biquandle brackets. *Journal of Knot Theory and Its Ramifications*, 26(05):1750034, 2017.

[2] Mikhail Khovanov. A categorification of the jones polynomial. *Duke Mathematical Journal*, 101(3):359–426, 2000.