Research Notes

Noah Donald

Summer 2019

1 Voltage Graphs

1.1 Introduction

Let $G = (V, E)$ be a simple graph with vertices enumerated by $\{1, ..., n\}$ where $|V| = n$ for some $n \in \mathbb{N}$ and an unordered edge set $E \subseteq V \times V$ [1]. Let $\mathcal{H}$ be a group. Let L be a labeling of the edges of G by elements of the group $\mathcal{H}$. In other words, $L : E \mapsto \mathcal{H}$. Let κ_1 be a vertex coloring of the graph G with natural numbers and let κ_2 be a vertex coloring of the graph G with elements from the group $\mathcal{H}$. In other words, $\kappa_1 : V \mapsto \mathbb{N}$ and $\kappa_2 : V \mapsto \mathcal{H}$. Let $\kappa = (\kappa_1, \kappa_2)$ map $V \mapsto \mathbb{N} \times \mathcal{H}$. We will also want to associate a direction or "polarization" to each edge of G. It will be useful to represent the polarization of each edge, $(v_i, v_j) \in E$, by the function $\epsilon : E \mapsto \{-1, 1\}$ which is defined using the following convention:

$$\epsilon((v_i, v_j)) = \begin{cases} 1 & v_i \to v_j \\ -1 & v_j \to v_i \end{cases}$$

Hence, a choice of the function ϵ amounts to choosing a polarization of the edges of the graph. Notice also that $\epsilon((v_i, v_j)) = -\epsilon((v_j, v_i))$. Thus, the function is anti-symmetric. This system, $(G, \mathcal{H}, L, \epsilon)$, is called a voltage graph.

1.2 Proper Colorings: A Definition

To define a proper coloring of an edge of a voltage graph, we consider an edge, $e = (v_i, v_j)$, a vertex coloring, κ, and an edge labeling L. A coloring of this graph is said to be proper if any one of the following conditions holds:

1. $\kappa_1(v_i) = \kappa_1(v_j)$ and if $\epsilon(e) = 1$, then $L(e)\kappa_2(v_i) \neq \kappa_2(v_j)$.
2. $\kappa_1(v_i) = \kappa_1(v_j)$ and if $\epsilon(e) = -1$, then $\kappa_2(v_i) \neq L(e)\kappa_2(v_j)$.
3. $\kappa_1(v_i) \neq \kappa_1(v_j)$.

[1] One could also use an ordered edge set where $(v_i, v_j) \in E$ implies that the edge is oriented from vertex v_i to vertex v_j. The upshot of using this definition is that most of the formulas simplify, however the disadvantage is that the effect of the edge orientation does not become evident in the formulas. I chose this definition because it will prove useful later to see how the edge orientation enters formulas.

1.3 Proper Colorings: Finding a Formula

We want to encode the proper coloring condition into a mathematical formula. I will now discuss a way to go about quantifying these conditions using the functions defined in the introduction.

First, assume that for a graph, $G = (V, E)$, the edge labeling, L, is fixed. Let δ_{ij} be the Kronecker-delta operator and consider an edge, $e = (v_i, v_j)$, and a vertex coloring, κ. Let,

$$\tilde{\delta}(e) = \begin{cases} \delta_{L(e)\kappa_2(v_i),\kappa_2(v_j)} & \epsilon(e) = 1 \\ \delta_{\kappa_2(v_i),L(e)\kappa_2(v_j)} & \epsilon(e) = -1 \end{cases}$$

Now consider the function $f : E \mapsto \{0, 1\}$ defined by

$$f(e) = 1 - (\frac{1}{2})((\delta_{\kappa_1(v_i),\kappa_1(v_j)} + \tilde{\delta}(e)) - |\delta_{\kappa_1(v_i),\kappa_1(v_j)} - \tilde{\delta}(e)|)$$

.
Note that this function returns the value 1 if the edge is colored properly and the value 0 if the edge is colored improperly. We can find a formula for $\tilde{\delta}$ by setting,

$$\tilde{\delta}(e) = (\frac{1}{2})(\delta_{L(e)\kappa_2(v_i),\kappa_2(v_j)}(1 + \epsilon(e)) + \delta_{\kappa_2(v_i),L(e)\kappa_2(v_j)}(1 - \epsilon(e))).$$

We can also replace the term containing an absolute value sign in the formula for f by setting,

$$|\delta_{\kappa_1(v_i),\kappa_1(v_j)} - \tilde{\delta}(e)| = \delta_{\kappa_1(v_i),\kappa_1(v_j)} + \tilde{\delta}(e) - 2\delta_{\kappa_1(v_i),\kappa_1(v_j)}\tilde{\delta}(e).$$

After making these substitutions into the function, f, and simplifying the expression, we are left with the following formula:

$$f(e) = 1 - \frac{1}{2}\delta_{\kappa_1(v_i),\kappa_1(v_j)}(\delta_{L(e)\kappa_2(v_i),\kappa_2(v_j)} + \delta_{\kappa_2(v_i),L(e)\kappa_2(v_j)} + \epsilon(e)(\delta_{L(e)\kappa_2(v_i),\kappa_2(v_j)} - \delta_{\kappa_2(v_i),L(e)\kappa_2(v_j)})).$$

I will now introduce some new notation to help clean some things up. Let,

$$\epsilon((v_i, v_j)) = \epsilon_{ij},\ \delta_{\kappa_1(v_i),\kappa_1(v_j)} = \delta_{ij},\ \delta_{L(e)\kappa_2(v_i),\kappa_2(v_j)} = \delta^{Lij}, \delta_{\kappa_2(v_i)L(e)\kappa_2(v_j)} = \delta^{iLj},\ L((v_i, v_j)) = L^{ij}$$

Notice that subscripts of an operator denote a labeling of a vertex or edge in $\mathbb{N}$ and superscripts of an operator denote a labeling of a vertex in $\mathcal{H}$. In the superscripts of the Kronecker-delta operator, I dropped the superscript on the symmetric operator, L, since it should be apparent that L is acting on the group elements assigned to v_i and v_j.

Let us now define the operator $\phi_{ij} = \frac{1}{2}\delta_{ij}(\delta^{Lij} + \delta^{iLj} + \epsilon_{ij}(\delta^{Lij} - \delta^{iLj}))$[2]. Then the formula for f simplifies to $f(e) = 1 - \phi_{ij}$. It is important to note that ϕ_{ij} is a symmetric operator, since δ_{ij} and $(\delta^{Lij} + \delta^{iLj})$ are both symmetric operators, while $\epsilon_{ij}(\delta^{Lij} - \delta^{iLj})$ is a combination of two anti-symmetric operators which makes it symmetric.

[2]In the alternative definition of the voltage graph mentioned in the footnote in the introduction section, the proper coloring condition formula simplifies to $\delta_{ij}\delta^{Lij}$.

1.4 A Symmetric Chromatic Function

We want to create a symmetric chromatic polynomial for a voltage graph so that the coefficients of the polynomial carry a combinatorial interpretation. For this function I will fix the graph, G, the edge labeling, L, and the edge polarization, ϵ. With these functions fixed, the polynomial will be free to range over all possible vertex colorings, κ. I also want to make one more notational adjustment to keep the clutter in the formula to a minimum. Let $(v_i, v_j) = (i, j)$ which should be somewhat natural since edges are the only place where a 2-tuple is considered in this document with the exception of the definition of κ, but I will explicitly state when κ is being used. I will define the symmetric chromatic function, $\chi_{(G,\mathcal{H},L,\epsilon)}$, by,

$$\chi_{(G,\mathcal{H},L,\epsilon)} = \sum_{\kappa} x_\kappa \prod_{(i,j)\in E} (1 - \phi_{ij}).$$

Here the sum is over all maps $\kappa : V \mapsto \mathbb{N} \times \mathcal{H}$ and the symbol x_κ is shorthand for $x_\kappa = x_{\kappa(v_1)} x_{\kappa(v_2)} ... x_{\kappa(v_n)}$. Lets first consider the product component of this formula. We know that the product, $\prod_{(i,j)\in E}(1 - \phi_{ij})$, is equivalent to a sum of terms involving ϕ_{ij}. In fact, each term in the sum is equivalent to choosing either 1 or $-\phi_{ij}$ for each $(i, j) \in E$ and then taking the product of the chosen terms. But, this is equivalent to just choosing a subset of edges from E and then forming their product. Hence we have the following expansion of the product:

$$\prod_{(i,j)\in E} (1 - \phi_{ij}) = \sum_{S\subseteq E} (-1)^{|S|} \prod_{(i,j)\in S} \phi_{ij}.$$

Now lets focus for a moment on the new product term, $\prod_{(i,j)\in S} \phi_{ij}$. Expanding out ϕ_{ij} by its definition yields,

$$\prod_{(i,j)\in S} \phi_{ij} = \frac{1}{2^{|S|}} \prod_{(i,j)\in S} \delta_{ij}(\delta^{Lij} + \delta^{iLj} + \epsilon_{ij}(\delta^{Lij} - \delta^{iLj}).$$

It will be important to figure out when a term $S \subseteq E$ in the sum is not destroyed by the product term. It is clear from the formula that for all $e = (i, j) \in S$ we need $\kappa_1(i) = \kappa_1(j)$. With that said, lets decompose S into its connected components. Suppose $S = \bigsqcup_{i\in\{1,...,l\}} \mathcal{S}_i$ for some $l \in \mathbb{N}$ where $(\bigsqcup)$ denotes the disjoint union operation and each $\mathcal{S}_i$ is a connected component of S. Then for each $i \in \{1, ..., l\}$, for all $v_j \in \mathcal{S}_i$, we have $\kappa_1(v_j) = k$ for some $k \in \mathbb{N}$. In other words, each connected component needs to be colored the same in the natural numbers for the term to have a chance of surviving in the sum.

Lets now consider that other term in the product, $\delta^{Lij} + \delta^{iLj} + \epsilon_{ij}(\delta^{Lij} - \delta^{iLj})$. For the sake of convenience I will call this term γ_{ij}. We wish to find the necessary conditions for which $\gamma_{ij} = \delta^{Lij} + \delta^{iLj} + \epsilon_{ij}(\delta^{Lij} - \delta^{iLj}) \neq 0$. We will approach this problem in a case by case fashion and since we are dealing with a product we need only consider single $e = (i, j) \in S$. First, suppose $\delta^{Lij} = \delta^{iLj} = 1$. Then $\gamma_{ij} = 2$ and note that this case is entirely independent of the value of ϵ_{ij}. Next, suppose $\delta^{Lij} = \delta^{iLj} = 0$. Then $\gamma_{ij} = 0$ which would nullify that subset of edges. Finally, suppose $\delta^{Lij} \neq \delta^{iLj}$. Then $\gamma_{ij} = (1 + \epsilon_{ij}(\delta^{Lij} - \delta^{iLj}))$. If $\delta^{Lij} = 0$, then we must have $\epsilon_{ij} = -1$ for $\gamma_{ij} \neq 0$. Likewise, if $\delta^{iLj} = 0$, then we must have $\epsilon_{ij} = 1$ for $\gamma_{ij} \neq 0$. In words, these cases imply that for some $e = (i, j) \in S$, $\gamma_{ij} \neq 0$ if and only if the group element of one vertex can be transformed by L^{ij} into the group element of the other vertex in the direction of the polarization. With that said, lets again decompose S into its connected components, $S = \bigsqcup_{i\in\{1,...,l\}} \mathcal{S}_i$. In this case however, I want to consider for each $i \in \{1, ..., l\}$, the set of directed paths contained in $\mathcal{S}_i$. Lets call this set $P_i = \{[(j,k)]_{(j,k)\in\mathcal{S}_i} : \epsilon_{jk} = \epsilon_{km} = ... = \epsilon_{nl} = 1, l \in \mathbb{N}\}$. Here the bracket notation for the paths just signifies that we are dealing with an ordered set. I also want to introduce the map $L_i : P_i \mapsto \mathcal{H}$ defined by $L_i(p) = L^{rj}L^{jk}...L^{lm}$ where $p = [(l,m), ..., (j,k), (r,j)]$. The $\gamma_{ij} \neq 0$ condition means that for each $i \in \{1, ..., l\}$, for each directed path, $p = [(j,m),(m,n), ..., (r,k)] \in P_i$, if v_j is the initial vertex in the path and v_k is the terminal vertex in the path, then there exists a group element, $L_i(p) = L^{rk}...L^{mn}L^{mj} \in \mathcal{H}$, such that $L_i(p)\kappa_2(v_j) = \kappa_2(v_k)$. In other words, any two vertices which are connected along a directed path, p, have group colorings such that the group coloring of the final vertex of the path is obtainable by the group coloring of the initial vertex in the path via the group element $L_i(p)$.

With these two necessary and sufficient conditions, we claim that a set $S \subseteq E$ survives in the sum if and only if the vertices contained in each connected component of S are colored in the natural numbers identically and for each connected component of S, for each path, p in the set of all directed paths in a component, the group labeling of the terminal vertex is obtainable from the group labeling of the initial vertex by $L_i(p)$. These conditions should be somewhat intuitive since they are precisely the conditions necessary for a non-proper coloring to occur on one edge, however now that notion is extended to the connected components of a subset of the edges of the voltage graph.

With these conditions in mind, lets now return to our definition of the symmetric chromatic function.

$$\chi_{(G,\mathcal{H},L,\epsilon)} = \sum_{\kappa} x_\kappa \prod_{(i,j)\in E} (1 - \phi_{ij}).$$

Substituting in the formulas which expand the product term give,

$$\chi_{(G,\mathcal{H},L,\epsilon)} = \sum_{\kappa} x_\kappa \sum_{S\subseteq E} (\frac{-1}{2})^{|S|} \prod_{(i,j)\in S} \delta_{ij}(\delta^{Lij} + \delta^{iLj} + \epsilon_{ij}(\delta^{Lij} - \delta^{iLj}).$$

To check this formula and explore some of its properties, lets check that it still yields the value 1 when κ is a proper coloring and 0 otherwise. Lets suppose we have a coloring $\kappa : V \mapsto \mathbb{N} \times \mathcal{H}$. Let $\mathcal{E}_\kappa \subseteq E$ represent the largest subset of edges of E on which κ satisfies a non-proper coloring. Then there are $2^{|\mathcal{E}_\kappa|}$ subsets of this set on which κ satisfies a non-proper coloring. Because we are dealing with the largest subset of edges of E on which κ satisfies the non-proper coloring condition, any nonempty subset of $E/\mathcal{E}_\kappa$ must be a subset of E on which κ is a proper coloring. Moreover, any nonempty subset of E which contains edges in both $\mathcal{E}_\kappa$ and $E/\mathcal{E}_\kappa$ also evaluates to zero in the product. Thus, the only subsets of E which survive in the sum are the subsets of $\mathcal{E}_\kappa$. The contribution in the sum of these terms is,

$$\sum_{S\subseteq\mathcal{E}_\kappa} (-1)^{|S|} = \sum_{k=0}^{k=|\mathcal{E}_\kappa|} (-1)^k \binom{|\mathcal{E}_\kappa|}{k} = 0.$$

Thus, the only time one of these terms contributes to the sum is when $|\mathcal{E}_\kappa| = 0$. In other words, a subset only contributes to the sum when it is a proper coloring on the entire edge set, E. In this case,

$$\sum_{k=0}^{k=0} (-1)^k \binom{0}{k} = 1.$$

This evaluation of the sum verifies that our algebraic manipulations so far have been valid.

We can also interchange the sum over κ and the sum over $S \subseteq E$ which shows that,

$$\chi_{(G,\mathcal{H},L,\epsilon)} = \sum_{S\subseteq E} (\frac{-1}{2})^{|S|} \sum_{\kappa} x_\kappa \prod_{(i,j)\in S} \delta_{ij}(\delta^{Lij} + \delta^{iLj} + \epsilon_{ij}(\delta^{Lij} - \delta^{iLj}).$$

This form of the symmetric chromatic function provides a different interpretation of the symmetric chromatic function and an evaluation of how it works is similar in nature to the one given in the previous paragraph. With all of this said, it should be clear that if we have some term x_κ in our symmetric chromatic function for a voltage graph, then the coefficient of this term is the number of colorings of the vertices of the voltage graph using only and all of the elements of the set $Im[\kappa] \subseteq \mathbb{N} \times \mathcal{H}$ with correct multiplicities. Here $Im[\kappa]$ is the image of the function κ on the set V. This gives the combinatorial interpretation of the coefficients of the symmetric chromatic function eluded to earlier.

1.5 Deletion of Edges and Transforming Full Edges into Squiggly Edges

If we return to the original, more succinct, formulation of the symmetric chromatic function, we can find an operation that allows us to manipulate the voltage graph, yet leave the symmetric chromatic function invariant. To introduce this operation, lets suppose we single out an edge $(\alpha, \beta) \in E$ as part of our voltage graph $(G = (V, E), \mathcal{H}, L, \epsilon)$. Then,

$$\chi_{(G=(V,E),\mathcal{H},L,\epsilon)} = \sum_{\kappa} x_\kappa \prod_{(i,j)\in E/(\alpha,\beta)} (1 - \phi_{ij})(1 - \phi_{\alpha\beta})$$

$$= \sum_{\kappa} x_\kappa \prod_{(i,j)\in E/(\alpha,\beta)} (1 - \phi_{ij}) - \sum_{\kappa} x_\kappa \prod_{(i,j)\in E/(\alpha,\beta)} (1 - \phi_{ij})\phi_{\alpha\beta}$$

The first term here represents the voltage graph with the the edge (α, β) deleted, while the second term is the same voltage graph as before, but now the edge (α, β) must satisfy the non-proper coloring condition discussed earlier. We will call this new type of edge which needs to satisfy the non-proper coloring condition a squiggly edge. Thus, we can decompose the edge set into a squiggly edge set and a full edge set, so $E = F \sqcup S$.

To write this explicitly in terms of the symmetric chromatic function, suppose we have a graph, $G = (V, E)$. Let $F \subseteq E$ be the set of full edges and $S \subseteq E$ be the set of squiggly-edges. Then $E = F \sqcup S$. Let $e \in F$. Then,

$$\chi_{((V,F),\mathcal{H},L,\epsilon)} = \chi_{((V,(F/e)\sqcup S),\mathcal{H},L,\epsilon)} - \chi_{((V,(F/e)\sqcup(S\cup e)),\mathcal{H},L,\epsilon)}.$$

In words, the symmetric chromatic function of a graph is the same as the symmetric chromatic function of the same graph with some edge deleted minus the symmetric chromatic function of the same graph, but with that edge now transformed into a squiggly-edge. This formula is known as the deletion-squiggly formula. Notice how the deletion-squiggly formula is useful. It can be applied in iteration to a graph to produce terms involving only disconnected vertices and vertices connected by squiggly-edges.

1.6 Squiggly-Graphs

It remains then to investigate graphs connected solely by squiggly-edges. Suppose we have a connected tree consisting solely of squiggly-edges. Cycles will be discussed later, so for now we will restrict our attention to trees. In order for the tree to be properly colored, all of the vertices must be colored identically in the natural numbers. We must also have that any two vertices which are connected along a directed path, p, have group colorings such that the group coloring of the final vertex in the path is obtainable by group coloring of the initial vertex in the path via the group element $L(p)$. Here I dropped the "i" subscript since this squiggly-tree is connected.

There are two types of vertices which are of particular interest in this connected squiggly-tree. The first type will be identified as a "source" and the second type will be identified as a "sink". A vertex, $v_i \in V$, is a source if for each edge of the form (i, j), $\epsilon_{ij} = 1$. In other words, a vertex is a source if for every edge connected to the vertex, the polarization of the edge is away from the vertex. Similarly a vertex, $v_i \in V$, is a sink if for each edge of the form (i, j), $\epsilon_{ij} = -1$. In other words, a vertex is a sink if for every edge connected to the vertex, the polarization of the edge is towards the vertex. Let $\mathcal{I} = \{v \in V : v \text{ is a source vertex}\}$ and $\mathcal{T} = \{v \in V : v \text{ is a sink vertex}\}$. Lets also consider the set of directed paths whose initial vertex is a source vertex and terminal vertex is a sink vertex. We will call this set $\mathcal{L}[G]$. Note that each of these paths are of maximal length, in that any of these paths cannot be extended to a larger directed path past the sink vertex. The motivation for considering these vertices is that they act like the respective inputs and outputs of a system where the group elements assigned to the edges act like "transformations" that occur between pieces of the system.

Before constructing the corresponding sets of inputs and outputs of the system, it will be important to introduce another operation on the edges of a voltage graph. The symmetric chromatic function remains invariant if we flip the polarization assigned to any edge (i, j) of the voltage graph as well as change $L^{ij} \to (L^{ij})^{-1}$. This operation allows us to change sources into sinks as well as reverse the direction of paths without changing any combinatorial properties of the graph.

To construct all successful combinations of inputs for this squiggly-tree, lets look at just one of the vertices, v_i. Because this tree is connected, there exists a unique path (not yet directed) from v_i to any other vertex in the graph. Suppose we color v_i with the group element g. Then by using the just discussed operation on the edges of this voltage tree, we can form a directed path p_{ij} from vertex v_i to some other vertex, v_j. In order for a proper coloring of the voltage tree to occur, we need v_j to be colored in the group with the element $L(p_{ij})$. In this way we can color all of the vertices of the tree with group elements and obtain a proper coloring. In fact, by allowing the group element assigned to v_i to vary over the entire group, we can, in this way, generate all possible proper colorings of the squiggly voltage tree. Since the initial choice of vertex was arbitrary, for any vertex in the tree, for any element of the group, we can find a proper coloring of the squiggly-tree that uniquely assigns the chosen group element to the chosen vertex. Hence, the number of proper colorings of the squiggly voltage tree is the same as the cardinality of the group in this case. Using this method it is possible to construct the set of valid inputs into the system that produce consistent outputs.

1.7 Vortexes

A vortex is a cycle in a graph with all of the polarizations in the same direction, i.e. clockwise or counterclockwise. To analyze this substructure lets consider the smallest possible vortex, a squiggly-triangle with clockwise polarization. Suppose I label these three vertices clockwise as $\{v_1, v_2, v_3\}$ and pick any one of the three vertices to color with a group element $g \in \mathcal{H}$. If, for example, I chose v_1, then in order for a proper coloring of the graph to occur, v_2 must be colored with $L^{12}g$, v_3 must be colored with $L^{23}L^{12}g$, and v_1 must be colored with $L^{13}L^{23}L^{12}g$. But we already colored v_1 with the group element g, so we require that $L^{13}L^{23}L^{12}g = g$. Hence, we must have that $L^{13}L^{23}L^{12} = e$ where e is the identity element of the group. Notice how our choice group coloring of the initial vertex was superfluous in trying to find a proper coloring of the vortex. This implies that in order for any proper coloring of a graph containing a vortex to occur, we need the ordered composition of the edge colorings to be equivalent to the identity element.

As a possible direction going forward, it may be interesting to look at the set of closed paths or cycles in a general voltage graph. Each of these closed paths from a chosen base point (actually doesn't depend on choice of basepoint) represents an element of the group given by $L(p)$. We could then look at the subgroup generated by these closed paths. While this doesn't have much to do with the symmetric chromatic function since it is not colorable except in the case where $L(p) = e$ for all such closed paths, it may be a useful tool to study the structure of the group used in the voltage graph. This avenue of inquiry stems from a more dynamical motivation, since closed orbits in the graph represent a group action of this generated subgroup on the original group.

1.8 Moving Rules

Suppose I have a graph which contains both squiggly and full edges in it. I want to first analyze what happens in terms of the colorings when I move a squiggly edge over a squiggly edge in the graph. This in essence changes the shape of the graph and allows us to explore algebraic bases of the symmetric chromatic function. I will also look at moving full edges over squiggly edges.

Our first goal is to move a squiggly edge over a squiggly edge without changing the symmetric chromatic function of the voltage graph. Suppose I have three vertices, $\{v_1, v_2, v_3\}$ and squiggly edges $\{(1,2),(2,3)\}$ where $\epsilon_{12} = 1$ and $\epsilon_{23} = 1$. I want to compare the colorings of this original graph with the possible colorings of the transformed graph consisting of the same vertex set, but $(1,2) \to (1,3)$. In the original graph we require that all of the vertices be colored by the same natural number. The same holds true for the transformed graph since it is still a connected graph of squiggly edges. Now in the original graph, if we color vertex v_1 with a group element, h_1, then v_2 is colored with $L^{12}h_1$. Likewise v_3 is colored with $L^{23}L^{12}h_1$. We need the same coloring to occur in the transformed graph for the symmetric chromatic function to remain invariant under the transformation. In the transformed graph, if we once again color the vertex, v_1, with the group element, h_1, the group coloring for v_3 must be $L^{13}h_1$. However, we must have $L^{13}h_1 = L^{23}L^{12}h_1$ or after simplifying $L^{13} = L^{23}L^{12}$. In other words, the moving edge $(1,2)$ picks up a "phase" from the edge it moved over. In this case, the phase is L^{23}. We also have that v_2 is colored with $(L^{23})^{-1}L^{13} = (L^{23})^{-1}L^{23}L^{12} = L^{12}$ which agrees with the original graph. If we are trying to move over squiggly edges with different polarizations than the ones used in this example, simply change the polarizations to agree with the ones discussed here and change the edge labeling from L^{ij} to $(L^{ij})^{-1}$. The conditions for moving over edges in these cases remain the same.

Now lets look at moving a full edge over a squiggly edge. Using a similar setup to the previous case of moving a squiggly edge over a squiggly edge, we have a vertex set $\{v_1, v_2, v_3\}$, full edge set $\{(1, 2)\}$, and squiggly edge set $\{(2, 3)\}$ with $\epsilon_{12} = \epsilon_{23} = 1$. For both the original graph and the transformed graph we need vertices v_2 and v_3 to be colored identically in the natural numbers. If v_1 is colored differently in the natural numbers than v_2 and v_3 then we have a proper coloring in both the original graph and the transformed graph. Suppose then that v_1 is colored identically in the natural numbers to v_2 and v_3. Then if we color v_1 with the group element h_1 and v_2 with the group element h_2, we need $L^{12}h_1 \neq h_2$ and v_3 to be colored with $h_3 = L^{23}h_2$. This implies that $L^{23}L^{12}h_1 \neq h_3$. In the transformed graph we need $L^{13}h_1 \neq h_3$ and $h_2 = (L^{23})^{-1}h_3$. We can see that $L^{13} = L^{23}L^{12}$ and the other condition for the squiggly edge is satisfied automatically. The condition is equivalent to the one found for moving a squiggly edge over another squiggle edge.

We can actually find a general formula for L^{13} in both the case of moving a squiggly edge over a squiggly edge and moving a full edge over a squiggly edge. The formula is given by $(L^{13})^{\epsilon_{13}} = (L^{23})^{\epsilon_{23}}(L^{12})^{\epsilon_{12}}$ where the vertex labeling and voltage graph assumed here is the same as the one used in the two situations discussed in this section.

As a possible direction going forward, lets suppose we have a squiggly connected voltage tree where the edges are labeled with elements of a group. We can replace this voltage tree with a voltage star. A voltage star on $n + 1$ vertices is a connected voltage graph where a single vertex has degree n and all other vertices have degree equal to 1. We can also choose this voltage star to be "radiating" which means that the vertex of degree n is a source vertex of the tree. By taking the squiggly leaf edges of this voltage tree and moving them to the chosen vertex we can iteratively create a radiating voltage star with the same symmetric chromatic function as the original voltage tree. We can now apply the reverse process using the deletion-squiggly formula to this star to unsquiggly this radiating voltage star into a set full edge radiating voltage stars. This process can be applied more generally to voltage graphs to get a radiating voltage star decomposition of the symmetric chromatic function. It may also be useful to take a look at other algebraic bases such as the path basis. It may prove fruitful to see if the terms present in the voltage star or voltage path bases provide any kind of combinatorial insight into the structure of our original graph.