A Symmetric Chromatic Function for Signed Graphs

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Chromatic Polynomial

Proper Colorings

Given a graph G a *proper coloring* of G by A is a function $f : G \to A$ such that for all pairs of adjacent vertices v, w, $f(v) \neq f(w)$.

Chromatic Polynomial

Given a graph G, define the *chromatic polynomial* by $\chi_G(q) =$ the number of proper colorings of G by $\{1, 2, \ldots, q\}$.

Symmetric Chromatic Function

Symmetric Chromatic Function (Stanley, 1995)

$$X_G(x_1, x_2, \ldots) := \sum_{\substack{\kappa : V(G) \to \mathbb{N} \\ \text{proper}}} \prod_{v \in V(G)} x_{\kappa(v)}$$

Alternatively,

$$X_G(x_1, x_2, \ldots) = \sum_{\substack{\kappa : V(G) \to \mathbb{N} \\ \text{all}}} \prod_{v \in V(G)} x_{\kappa(v)} \prod_{e=(v_1, v_2) \in E(G)} (1 - \delta_{\kappa(v_1), \kappa(v_2)})$$

X_G is invariant under permutations of the indices.

If $G =$ , then

$$X_G = \sum_{i,j,k \text{ distinct}} x_i x_j x_k + \sum_{i=k \neq j} x_i x_j x_k = \sum_{i,j,k \text{ distinct}} x_i x_j x_k + \sum_{i \neq k} x_i^2 x_k.$$

Signed Graph Colorings

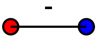
Signed Graph

A graph G and a function $\sigma : E(G) \to \{\pm\}$.

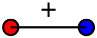
Signed Graph Coloring

A function $\kappa : V(G) \to \mathbb{Z} \setminus \{0\}$ or $\kappa : V(G) \to \{-n, \ldots, -1, 1, \ldots, n\}$ is a proper if for all adjacent vertices v, w, $\kappa(v) \neq \sigma(v, w)\kappa(w)$.

Assuming red = $-$blue,



is bad, while



is good.

Signed Chromatic Polynomial

Balanced Signed Chromatic Polynomial (Zaslavsky, 1982)

Given a signed graph G, the balanced signed chromatic polynomial satisfies $\chi_G^{\pm}(2n)$ = the number of proper colorings of G by $\{-n, \ldots, -1, 1, \ldots, n\}$.

Signed Symmetric Chromatic Function

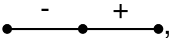
Signed Symmetric Chromatic Function

$$Y_G(\ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots) = \sum_{\substack{\kappa : V(G) \to \mathbb{Z} \setminus \{0\} \\ \text{proper}}} \prod_{v \in V(G)} x_{\kappa(v)}.$$

Alternatively,

$$Y_G = \sum_{\substack{\kappa : V(G) \to \mathbb{Z} \setminus \{0\} \\ \text{all}}} \prod_{v \in V(G)} x_{\kappa(v)} \prod_{e=(v_1,v_2) \in E(G)} (1 - \delta_{\kappa(v_1), \sigma(v_1,v_2)\kappa(v_2)}).$$

Y_G is invariant under *signed* permutations of the indices: permutations π satisfying $\pi(-k) = -\pi(k)$ for all k.

If $G =$ 

, then $Y_G = \sum_{-i,j,k \text{ distinct}} x_i x_j x_k + \sum_{-i=k \neq j} x_i x_j x_k.$

Power Sum Bases

Power Sum Symmetric Functions

$$P_n(x_1, x_2, \ldots) = \sum_{k \in \mathbb{N}} x_k^n.$$

Signed Power Sum Symmetric Functions

$$Q_{a,b}(\ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots) = \sum_{k \in \mathbb{Z} \setminus \{0\}} x_k^a x_{-k}^b$$

Applying Newton's identities to calculations from the previous slides give:

If $G =$ , then $X_G = P_3 - 2P_2P_1 + P_1^3$.

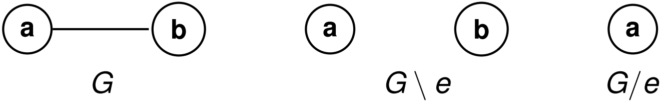
If $G =$ 

, then

$Y_G = Q_{1,0}^2 Q_{0,1} - Q_{2,0}Q_{0,1} - Q_{1,1}Q_{1,0} + Q_{2,1}$.

A Better Way to Compute?

Given a graph G and $(a, b) = e \in E(G)$, let $G \setminus e$ denote G with the edge G deleted and G/e denote the graph given by identifying the vertices a and b.



Theorem

$\chi_G = \chi_{G\setminus e} - \chi_{G/e}$.

X_G is homogeneous of degree $V(G)$, so this identity isn't true for X_G. By introducing weights, we can fix this.

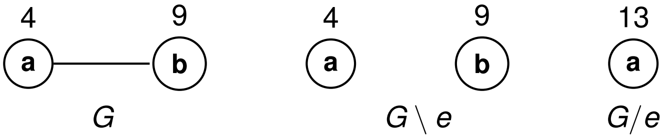
X_G for weighted graphs

Symmetric Chromatic Function for Weighted Graphs

Let G be a graph and let $\varphi : V(G) \to \mathbb{N}$ be a weighting of the vertices. Then, define $X_G(x_1, x_2, \ldots) := \sum_{\substack{\kappa : V(G) \to \mathbb{N} \\ \text{proper}}} \prod_{v \in V(G)} x_{\kappa(v)}^{\varphi(v)}$.

Weighted Contraction

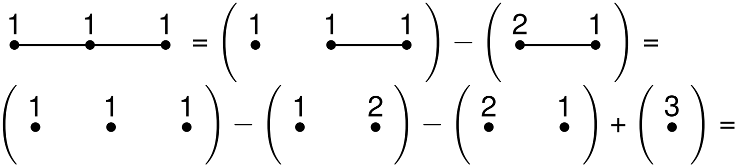
For a weighted graph G with weight function φ and an edge $(a, b) = e$, let G/e denote the graph given by identifying a and b and assigning this combined vertex the weight $\varphi(a) + \varphi(b)$.



Weighted Contraction-Deletion for X_G

If G is a single vertex with weight n, then $X_G = P_n$.

Theorem (Chmutov, et al. (1994), Noble, Welsh (1999))
Given a weighted graph G, we have $X_G = X_{G \setminus e} - X_{G/e}$.



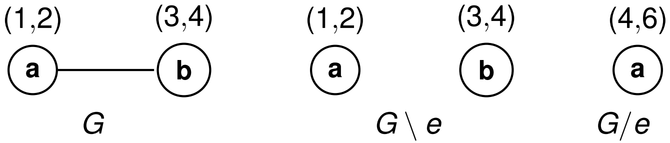
$P_1^3 - 2P_2P_1 + P_3$.

Y_G for Weighted Signed Graphs

Weighted Signed Graph

A weighted signed graph is a signed graph G and a function $\varphi : V(G) \to \mathbb{N} \cup \{0\} \times \mathbb{N} \cup \{0\}$.

Given a weighted signed graph G and an edge $(a, b) = e \in E(G)$, define G/e by identifying the vertices a and b and adding their weights coordinate wise.



Y_G for Weighted Signed Graphs

Y_G for Weighted Signed Graphs

Given a signed graph G with weight function $\varphi(v) = (\varphi_1(v), \varphi_2(v))$, define $Y_G = \sum_{\substack{\kappa: V(G) \to \mathbb{Z} \setminus \{0\} \\ \text{proper}}} \prod_{v \in V(G)} x_{\kappa(v)}^{\varphi_1(v)} x_{-\kappa(v)}^{\varphi_2(v)}$.

If G is a single vertex with weight (a, b), then $Y_G = Q_{a,b}$.

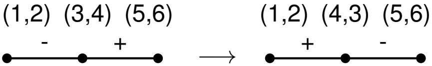
Weighted Contraction-Deletion for Y_G

Theorem

Let G be a weighted signed graph and $e \in E(G)$ be positive. Then, $Y_G = Y_{G \setminus e} - Y_{G/e}$.

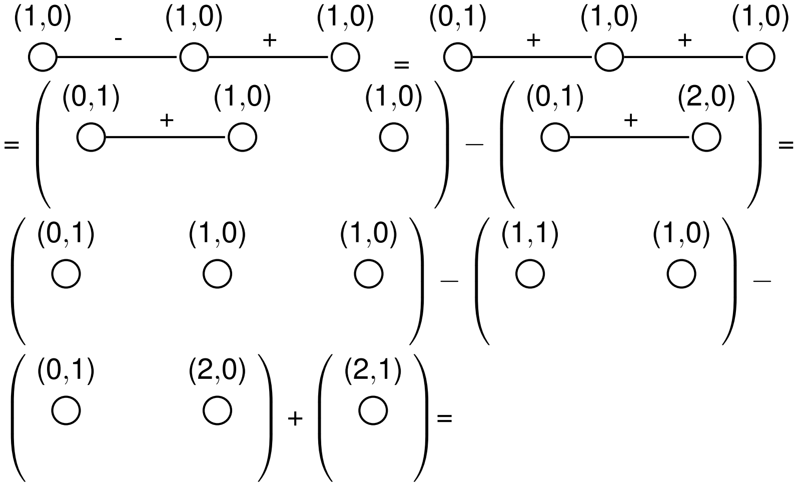
Theorem

Let G be a weighted signed graph, let v be a vertex with weight (a, b). Define the graph G' by reversing the signs of all edges incident to v and switching the weight of v to (b, a). Then, $Y_G = Y_{G'}$.



Calculating Y_G through Weighted Contraction-Deletion

Due to the switching identity, we need only ever perform contraction-deletion on positive edges.



$Q_{1,0}^2 Q_{0,1} - Q_{1,1} Q_{1,0} - Q_{2,0} Q_{0,1} + Q_{2,1}$.

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