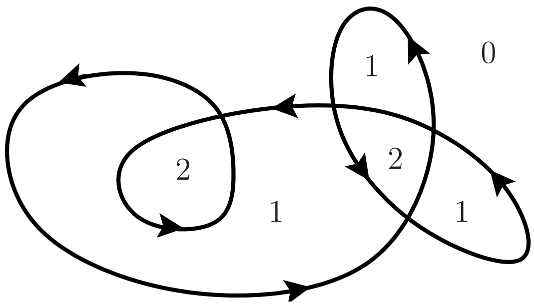
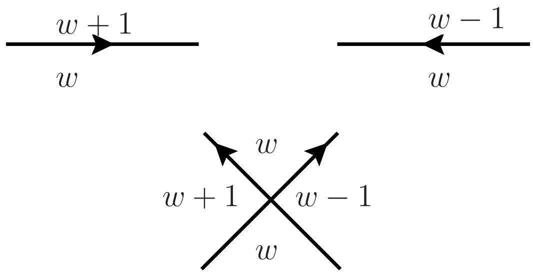
Plane curves.

A plane curve $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^2$ is *closed* if $\boldsymbol{\alpha}(a) = \boldsymbol{\alpha}(b)$. It is *immersed* if $\boldsymbol{\alpha}'(t) \neq 0$ for any $t \in [a, b]$.



Let $p \in \mathbb{R}^2$ be a point not on the curve $\boldsymbol{\alpha}$. The *winding number* $w_{\boldsymbol{\alpha}}(p)$ of an oriented closed curve $\boldsymbol{\alpha}$ around p is total number of (signed) turns made by $\boldsymbol{\alpha}$ around the point p.

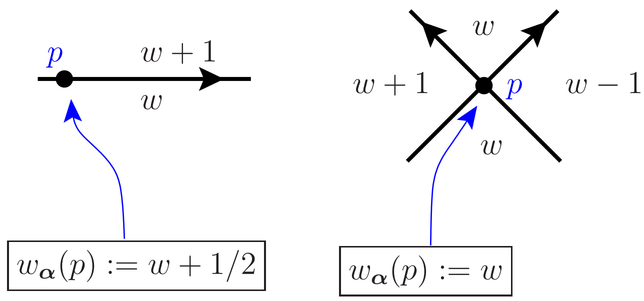


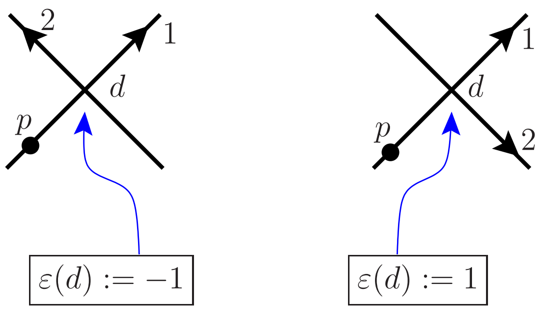
The winding number $w_{\boldsymbol{\alpha}}(p)$ changes when p goes across the curve as on the left figure. Thus it can be interpreted as the *intersection index* of $\boldsymbol{\alpha}$ with a path from p to ∞.

Fundamental Theorem of Algebra.
Any polynomial equation $f(z) = 0$ of degree > 1 has a (complex) solution.

Proof. We may assume that f has the highest coefficient 1 and $f(0) \neq 0$. Consider the family $\boldsymbol{\alpha}_R$ of plane curve given the image of the restriction of $f(z)$ on the circle $|z| = R$. For small values of R the curve $\boldsymbol{\alpha}_R$ is a small curve located near the value $f(0) \neq 0$. Thus its winding number around 0 is zero, $w_{\boldsymbol{\alpha}_R}(0) = 0$. When R is very large, the curve $\boldsymbol{\alpha}_R$ is very close to the curve $z^n\big|_{|z|=R}$ which has $w_{\boldsymbol{\alpha}_R}(0) = n$. That means for some intermediate values of R the winding number around 0 changes. It may happen only when the curve passes through 0. $\square$

A *generic* closed immersed curve $\boldsymbol{\alpha}$ has only finitely many transverse double points as singularities. For points p lying on $\boldsymbol{\alpha}$ define the winding number as the arithmetic mean of windings numbers of the incident regions as on the right figure. Fix a base point p distinct from the double points and assume that the parametrization of $\boldsymbol{\alpha}$ is such that $\boldsymbol{\alpha}(0) = p$.





For every double point d of $\boldsymbol{\alpha}$ we can define the sign $\varepsilon(d) = \pm 1$ in the following way. Let $t_1 < t_2$ be the two values of the parameter giving the double point: $\boldsymbol{\alpha}(t_1) = \boldsymbol{\alpha}(t_2) = d$. Then at d we get two vectors $\boldsymbol{\alpha}'(t_1)$ tangent to the first passage starting from p, and $\boldsymbol{\alpha}'(t_2)$ tangent to the second passage. Set $\varepsilon(d) := +1$ if the pair $(\boldsymbol{\alpha}'(t_1), \boldsymbol{\alpha}'(t_2))$ gives the negative orientation of $\mathbb{R}^2$. Otherwise set $\varepsilon(d) := -1$, see the left figure.

The *rotation index* $r_{\boldsymbol{\alpha}}$ of an immersed plane curve $\boldsymbol{\alpha}$ is the winding number of its derivative $\boldsymbol{\alpha}'$ around zero, $r_{\boldsymbol{\alpha}} := w_{\boldsymbol{\alpha}'}(0)$. There is Whitney's formula for the rotation index

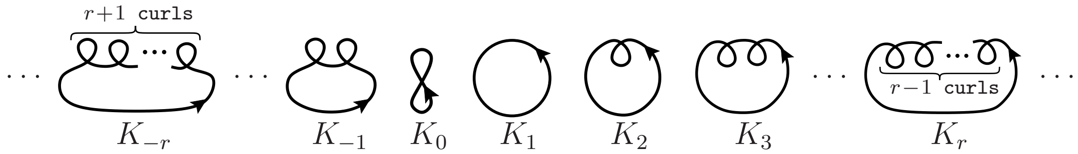
$$r_{\boldsymbol{\alpha}} = 2w_{\boldsymbol{\alpha}}(p) + \sum_d \varepsilon(d) \ ,$$

where the summation runs over all double points d of $\boldsymbol{\alpha}$.

A *homotopy* is a smooth deformation of an immersed curve.

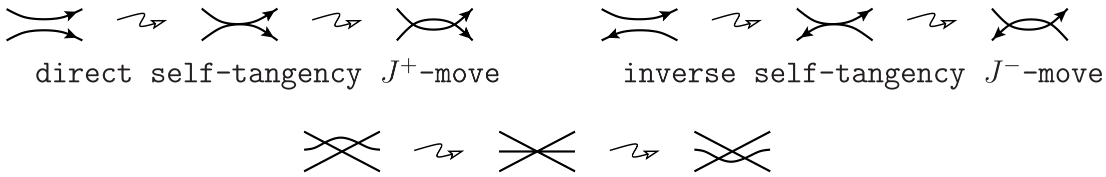
Whitney–Graustein Theorem.
The rotation index is the complete homotopy invariant of immersed plane curves.

This means that any two generic plane curves with the same rotation index can be deformed one to another in the class of immersed curved. We can choose one fixed curve for each values of the rotation index. Thus any plane immersed curve curve can be deformed to one the following *canonical curves*:



Canonical curves K_r with rotation index r

During a generic homotopy a finite number of the following moves may occur.



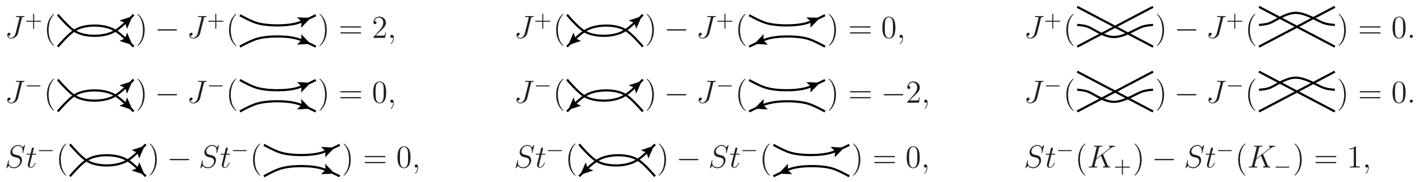
direct self-tangency J^+-move inverse self-tangency J^--move

triple point St-move

Arnold's invariants J^+, J^-, ***and*** St of immersed plane curves are defined by their initial values:

$$J^+(K_r) = \begin{cases} 0 & \text{for } r \leqslant 0 \\ -2(r-1) & \text{for } r > 0 \end{cases} \qquad J^-(K_r) = \begin{cases} r-1 & \text{for } r \leqslant 0 \\ -3(r-1) & \text{for } r > 0 \end{cases} \qquad St(K_r) = \begin{cases} 0 & \text{for } r \leqslant 0 \\ r-1 & \text{for } r > 0 \end{cases}$$

and the following skein relations:



where we define the subscript sign of K_+ and K_- at an St-move as follows. The three sides of the vanishing triangle at the St-move are directed as the whole curve is oriented. From the other hand, if we trace the curve we will pass the sided of the triangle in some definite cyclic order. This order induces another direction on the three sides. We define the sign to be equal to

$$(-1)^{\text{the number of sides on which the two directions are the same}} .$$