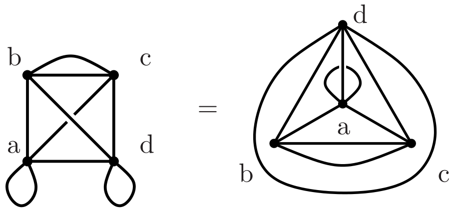
Graphs

Definition. A *graph* G is a finite set of *vertices* $V(G)$ and a finite set $E(G)$ of unordered pairs (x, y) of vertices $x, y \in V(G)$ called *edges*.
A graph may have *loops* (x, x) and *multiple edges* when a pair (x, y) appears in $E(G)$ several times. Pictorially we represent the vertices by points and edges by lines connecting the corresponding points. Topologically a graph is a 1-dimensional *cell complex* with $V(G)$ as the set of 0-cells and $E(G)$ as the set of 1-cells. Here are two pictures representing the same graph.



$V(G) = \{a, b, c, d\}$

$$E(G) = \{(a, a), (a, b), (a, c), (a, d), (b, c), (b, c), (b, d), (c, d), (d, d)\}$$

Chromatic polynomial $\chi_G(q)$.

A *coloring* of G with q colors is a map $c : V(G) \to \{1, \dots, q\}$. A coloring c is *proper* if for any edge e: $c(v_1) \neq c(v_2)$, where v_1 and v_2 are the endpoints of e.

Definition 1. *$\chi_G(q) :=$ # of proper colorings of G in q colors.*

Properties (Definition 2).
$\chi_G = \chi_{G-e} - \chi_{G/e}$;
$\chi_{G_1 \sqcup G_2} = \chi_{G_1} \cdot \chi_{G_2}$, for a disjoint union $G_1 \sqcup G_2$;
$\chi_\bullet = q$.

Stanley's theorem. *For a graph G with n vertices,*
$(-1)^n \chi_G(-1) =$ # of acyclic orientations of G.

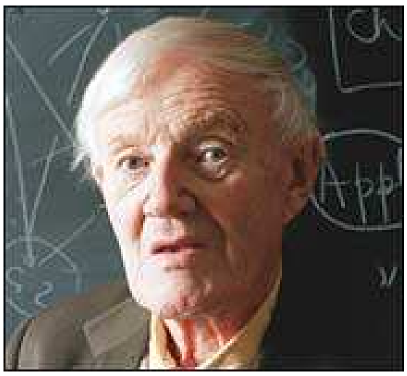
Flow polynomial $Q_G(q)$.

A *q-flow* on G is an assignment of a value $0, 1, \dots, q-1$ to every edge of G with arbitrarily chosen orientation of its edges in such a way that the total flow entering and leaving each vertex is congruent modulo q.

Definition 1. *$Q_G(q) :=$ # of nowhere-zero q-flows on G.*

Properties (Definition 2).

$Q_G(q) = 0$	if G has a bridge ;
$Q_G(q) = (q-1)Q_{G-e}(q)$	if e is a loop ;
$Q_G(q) = -Q_{G-e}(q) + Q_{G/e}$	if e is neither a bridge nor a loop ;
$Q_G(q) = \chi_{G^*}(q)/q$	for dual planar graphs G and G^* .



Tutte polynomial $T_G(x, y)$.

Definition 1.

$T_G = T_{G-e} + T_{G/e}$	*if e is neither a bridge nor a loop ;*
$T_G = xT_{G/e}$	*if e is a bridge ;*
$T_G = yT_{G-e}$	*if e is a loop ;*
$T_{G_1 \sqcup G_2} = T_{G_1 \cdot G_2} = T_{G_1} \cdot T_{G_2}$	*for a disjoint union $G_1 \sqcup G_2$ and a one-point join $G_1 \cdot G_2$;*

$T_\bullet = 1$.

Properties.

$T_G(1,1)$	is the number of spanning trees of G ;
$T_G(2,1)$	is the number of spanning forests of G ;
$T_G(1,2)$	is the number of spanning connected subgraphs of G ;
$T_G(2,2) = 2^{\lvert E(G)\rvert}$	is the number of spanning subgraphs of G ;

$\chi_G(q) = q^{k(G)}(-1)^{r(G)}T_G(1-q,0)$;
$Q_G(q) = (-1)^{n(G)}T_G(0,1-q)$.

Definition 2.

Let
- F be a graph;
- $v(F)$ be the number of its vertices;
- $e(F)$ be the number of its edges;
- $k(F)$ be the number of connected components of F;
- $r(F) := v(F) - k(F)$ be the *rank* of F;
- $n(F) := e(F) - r(F)$ be the *nullity* of F;

$$T_G(x,y) \ := \sum_{F\subseteq E(G)} (x-1)^{r(G)-r(F)}(y-1)^{n(F)}$$

Dichromatic polynomial $Z_G(q,v)$ **(Definition 3).**

Let $Col(G)$ denote the set of colorings of G with q colors.

$$Z_G(q,v) \ := \sum_{c\in Col(G)} (1+v)^{\#\text{ edges colored not properly by } c}$$

Properties .

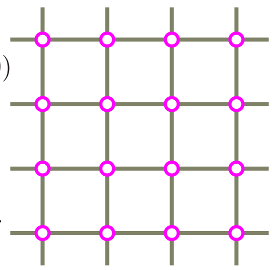
$Z_G = Z_{G-e} + vZ_{G/e}$;
$Z_{G_1\sqcup G_2} = Z_{G_1}\cdot Z_{G_2}$, for a disjoint union $G_1 \sqcup G_2$;
$Z_\bullet = q$;

$$Z_G(q,v) = \sum_{F\subseteq E(G)} q^{k(F)}v^{e(F)} \ ;$$

$\chi G(q) == Z_G(q,-1)$;

$Z_G(q,v) = q^{k(G)}v^{r(G)}T_G(1+qv^{-1}, 1+v)$;
$T_G(x,y) = (x-1)^{-k(G)}(y-1)^{-v(G)}Z_G((x-1)(y-1), y-1)$.

Potts model in statistical mechanics (Definition 4).



Potts model (C.Domb 1952); $q=2$ the Ising model (W.Lenz, 1920)

Let G be a graph.

Particles are located at vertices of G. Each particle has a *spin*, which takes q different values . A *state*, $\sigma \in \mathbb{S}$, is an assignment of spins to all vertices of G. Neighboring particles interact with each other only is their spins are the same.

The energy of the interaction along an edge e is $-J_e$ (*coupling constant*). The model is called *ferromagnetic* if $J_e > 0$ and *antiferromagnetic* if $J_e < 0$.

Energy of a state σ (*Hamiltonian*),

$$H(\sigma) = -\sum_{(a,b)=e\in E(G)} J_e\ \delta(\sigma(a),\sigma(b)).$$

Boltzmann weight of σ:

$$e^{-\beta H(\sigma)} = \prod_{(a,b)=e\in E(G)} e^{J_e\beta\delta(\sigma(a),\sigma(b))} = \prod_{(a,b)=e\in E(G)} \Big(1+(e^{J_e\beta}-1)\delta(\sigma(a),\sigma(b))\Big),$$

where the *inverse temperature* $\beta = \frac{1}{\kappa\ T}$, T is the temperature, $\kappa = 1.38\times 10^{-23}$ joules/Kelvin is the *Boltzmann constant*.

The Potts partition function (for $x_e := e^{J_e\beta}-1$)

$$Z_G(q,x_e)\ := \sum_{\sigma\in\mathcal{S}} e^{-\beta H(\sigma)} = \sum_{\sigma\in\mathcal{S}}\ \prod_{e\in E(G)} (1+x_e\delta(\sigma(a),\sigma(b)))$$

Properties of the Potts model Probability of a state σ: $P(\sigma) := e^{-\beta H(\sigma)}/Z_G$.

Expected value of a function $f(\sigma)$:

$$\langle f\rangle := \sum_\sigma f(\sigma)P(\sigma) = \sum_\sigma f(\sigma)e^{-\beta H(\sigma)}/Z_G\ .$$

Expected energy: $\langle H\rangle = \sum_\sigma H(\sigma)e^{-\beta H(\sigma)}/Z_G = -\frac{d}{d\beta}\ln Z_G$.

Fortuin—Kasteleyn'1972: $Z_G(q,x_e) = \sum_{F\subseteq E(G)} q^{k(F)}\prod_{e\in F} x_e$,

where $k(F)$ is the number of connected components of the spanning subgraph F.

$Z_G = Z_{G\setminus e} + x_e Z_{G/e}$.

Spanning tree generating function (Definition 5).

For a connected graph G fix an order of its edges: $e_1, e_2, \ldots, e_m$. Let T be a spanning tree.

An edge $e_i \in E(T)$ is called *internally active (live)* if $i<j$ for any edge e_j connecting the two components of $T-e_i$

An edge $e_j \notin E(T)$ is called *externally active (live)* if $j<i$ for any edge e_i in the unique cycle of $T\cup e_j$.

Let $i(T)$ and $j(T)$ be the numbers of internally and externally active edges correspondingly.

$$T_G(x,y)\ := \sum_T x^{i(T)}y^{j(T)}$$

Doubly weighted Tutte polynomial .

With each edge e of a graph G we associate two variables (weights) u_e and v_e.

$$T_G(\{u_e,v_e\},x,y) := \sum_{F\subseteq E(G)} \big(\prod_{e\in F} u_e\big)\ \big(\prod_{e\notin F} v_e\big)(x-1)^{r(G)-r(F)}(y-1)^{n(F)}$$

Properties.

$$\begin{array}{ll} T_G = v_e T_{G-e} + u_e T_{G/e} & \text{if } e \text{ is neither a bridge nor a loop ;} \\ T_G = (v_e(x-1)+u_e) T_{G/e} & \text{if } e \text{ is a bridge ;} \\ T_G = (v_e + (y-1)u_e) T_{G-e} & \text{if } e \text{ is a loop ;} \\ T_{G_1 \sqcup G_2} = T_{G_1 \cdot G_2} = T_{G_1} \cdot T_{G_2} & \text{for a disjoint union } G_1 \sqcup G_2 \\ & \text{and a one-point join } G_1 \cdot G_2 \text{ ;} \\ T_{\bullet} = 1 \ . & \end{array}$$

Tutte polynomial of signed graphs .

Signed graph is a a graphs with signs ± 1 assigned to the edges of the graph.

We define the Tutte polynomial of a signed graph by substituting the following weights to the doubly weighted Tutte polynomial.

$$+\text{-edge: } u_e := 1, \quad v_e := 1; \qquad -\text{-edge: } u_e := \sqrt{\frac{x-1}{y-1}}, \qquad v_e := \sqrt{\frac{y-1}{x-1}}.$$

With this substitution the Tutte polynomial for signed graphs becomes

$$T_G(x,y) = \sum_{F \subseteq E(G)} (x-1)^{r(G)-r(F)+s(F)} (y-1)^{n(F)-s(F)} \ ,$$

for $s(F) := \dfrac{e_-(F) - e_-(E(G) \setminus F)}{2}$, where $e_-(S)$ stands for the number of negative edges of S.

Chromatic polynomial of signed graphs.

There are two chromatic polynomials of signed graphs.

A q-*coloring* of a signed G is a map $c : V(G) \to \{-q, -q+1, \ldots, -1, 0, 1, \ldots, q-1, q\}$. A q-coloring c is *proper* if for any edge e with the sign ε_e: $c(v_1) \neq \varepsilon c(v_2)$, where v_1 and v_2 are the endpoints of e.

Definition.

$\chi_G(2q+1) :=$ *# of proper q-colorings of G.*

$\chi_G^{\neq 0}(2q) :=$ *# of proper q-colorings of G which take nonzero values.*

Properties.

- $\chi_G(\lambda)$ is a polynomial function of $\lambda = 2q+1 > 0$;
- $\chi_G^{\neq 0}(\lambda)$ is a polynomial function of $\lambda = 2q > 0$;
- $\chi_G(\lambda) = \chi_{G-e}(\lambda) - \chi_{G/e}(\lambda)$;
- $\chi_G^{\neq 0}(\lambda) = \chi_{G-e}^{\neq 0}(\lambda) - \chi_{G/e}^{\neq 0}(\lambda)$;
- $\chi_{G_1 \sqcup G_2} = \chi_{G_1} \cdot \chi_{G_2}$ and $\chi_{G_1 \sqcup G_2}^{\neq 0} = \chi_{G_1}^{\neq 0} \cdot \chi_{G_2}^{\neq 0}$ for a disjoint union $G_1 \sqcup G_2$;
- $\chi_{\emptyset} = 1$.