Covering Graphs and Linear Extensions of Signed Posets

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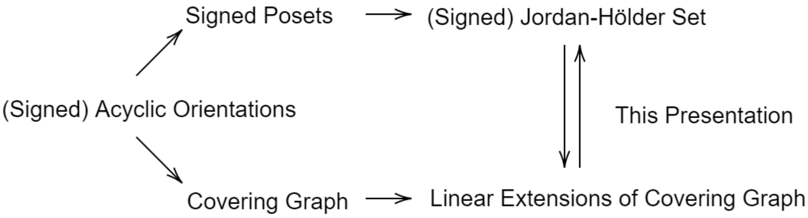
The Big Picture

Stanley's work in [1]:

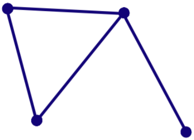
Acyclic Orientations ⟶ Posets ⟶ Linear Extensions = Jordan-Hölder Set

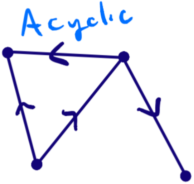
This presentation:

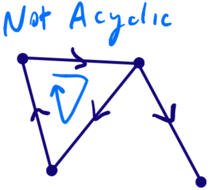


(Unsigned) Graphs and Orientations

A *graph* G is a set of vertices V with edges E connecting vertices. An *orientation* τ on the edges assigns each edge a direction. An orientation is *acyclic* if every cycle has a sink or a source.

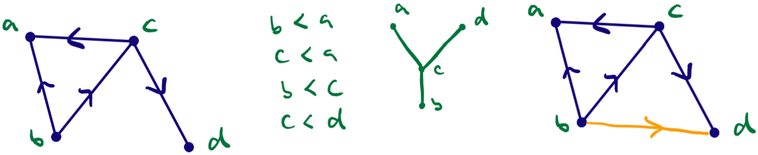






(Unsigned) Posets Associated to Acyclic Orientations

A *poset* P is a set with a partial order $<_P$ (written $<$ when there's no ambiguity). Partial orders are transitive and antisymmetric.
We can define a poset from an acyclic orientation τ by letting $u <_\tau v$ when an edge points from u to v.



(Unsigned) Linear Extension

Given a poset P, a *linear extension* P^* is a total order which preserves P. Namely for any $u \neq v \in P^*$:

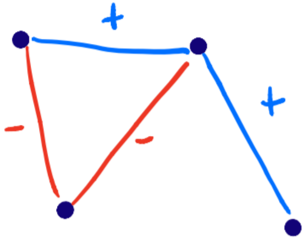
1. Either $u <_{P^*} v$ or $v <_{P^*} u$
2. If $u <_P v$, then $u <_{P^*} v$

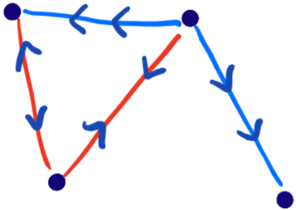
Signed Graphs and Orientations [Zaslavsky [2]]

A signed graph Σ is a graph where every edge is given a sign ± 1. An orientation τ now assigns to each *half edge* (i.e. the part of the edge next to a vertex) an arrow such that:

1. For positive edges the arrows face the same direction
2. For negative edges they face opposite directions

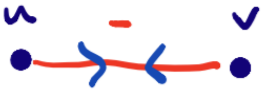
An orientation is acyclic if every cycle has a source or a sink.





Do Signed Acyclic Orientations define a Poset?

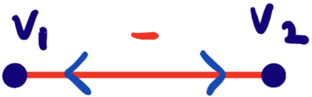
For positive edges, we have no issues. However, for negative edges, it's unclear which edge is bigger in the poset. For example, if the arrows on edge (u, v) point both into u and into v, then neither $u <_\tau v$ nor $v <_\tau u$. But then we're ignoring all of the negative edges from our poset!



The Root System Approach [Reiner [3]]

Instead of writing $v_i <_P v_j$, we use the vector $e_j - e_i$. If we have n elements in our poset, then these vectors live in $\mathbb{R}^n$.

For negative edges where both arrows point into the vertices, we then have no problem writing $e_j + e_i \in \mathbb{R}^n$.



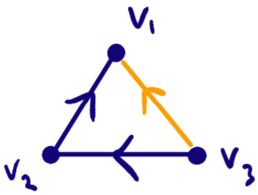
The A_n Root System

For (unsigned) posets, we use the A_n root system, $\Phi = \Phi^+ \cup -\Phi^+$, where

$$\Phi^+ = \{e_i - e_j \mid 1 \leq i < j \leq n\}$$

For the orientation shown below, the arrow pointing from v_2 to v_1 tells us $e_1 - e_2 \in P$. Furthermore,

$$P = \{\ e_1 - e_2,\ e_2 - e_3,\ e_1 - e_3\ \}$$



The B_n Root System

For signed posets, the vectors live in the B_n root system, $\Phi = \Phi^+ \cup -\Phi^+$, where

$$\begin{aligned}\Phi^+ = &\{e_1, e_2, \ldots, e_n\} \\ &\cup \{e_i - e_j \mid 1 \le i < j \le n\} \\ &\cup \{e_i + e_j \mid 1 \le i < j \le n\}\end{aligned}$$

Some examples of elements in Φ:

- $e_7, e_2 - e_4, \quad e_3 + e_{17} \in \Phi^+$
- $-e_7, e_4 - e_2, -e_3 - e_{17} \in -\Phi^+$.

Signed Posets

A subset $P^{\pm} \subseteq \Phi$ is a (signed) poset if it satisfies:

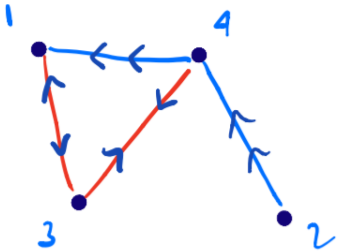
1. If $v \in P^{\pm}$ then $-v \notin P^{\pm}$
2. For $v, u \in P^{\pm}$ and $a, b \geq 0$, if $w = av + bu \in \Phi$, then $w \in P^{\pm}$

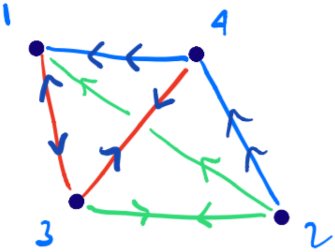
Signed Poset Example

In the orientation of graph below, the visible edges give us:

$$P^{\pm} = \{\ e_1 - e_4,\ e_1 + e_3,\ e_4 - e_2,\ -e_3 - e_4\ \}$$

Additionally, we have a *implied edges* $e_1 - e_2$ and $-e_3 - e_2$.





B-Symmetric Signed Permutations

A B-Symmetric signed permutation is a bijective function $\pi : \{-n, ..., n\} \setminus \{0\} \to \{-n, ..., n\} \setminus \{0\}$ such that $\pi(i) = -\pi(-i)$. Notice that this condition means it suffices to specify where the first n positive integers map to define the function.

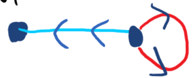
Example:

$$\pi = \begin{pmatrix} -2 & -1 & 1 & 2 \\ 1 & -2 & 2 & -1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

The Jordan-Hölder Set

Given a signed poset $P^{\pm}$ on n elements, the Jordan-Hölder set is denoted $\mathscr{L}(P^{\pm})$ where $\mathscr{L}(P^{\pm}) = \{\pi \in B_n : P^{\pm} \subseteq \pi\Phi^+\}$ where $\pi(e_i) = \text{sign}(\pi(i))e_{|\pi(i)|}$.

Jordan-Hölder Set Example



$$P^{\pm} = \{e_1 - e_2, -e_2\}$$

$$\Phi^+ = \{e_1, e_2, e_1 - e_2, e_1 + e_2\}$$

$$\pi_1 = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}$$

$$\pi_2 = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$$

$$\pi_1(e_1) = \text{sign}(\pi_1(1))\, e_{|\pi_1(1)|} = -e_2$$

$$\pi_1(e_2) = \text{sign}(\pi_1(2))\, e_{|\pi_1(2)|} = -e_1$$

$$\pi_1(e_1 - e_2) = \pi_1(e_1) - \pi_1(e_2) = e_1 - e_2$$

$$\pi_1(e_1 + e_2) = -e_1 - e_2$$

$$\pi_2(e_1) = \text{sign}(\pi_2(1))\, e_{|\pi_2(1)|} = e_1$$

$$\pi_2(e_2) = \text{sign}(\pi_2(2))\, e_{|\pi_2(2)|} = -e_2$$

$$\pi_2(e_1 - e_2) = \pi_2(e_1) - \pi_2(e_2) = e_1 + e_2$$

$$\pi_2(e_1 + e_2) = e_1 - e_2$$

$$\mathcal{L}(P^{\pm}) = \{\pi_1, \pi_2\}$$

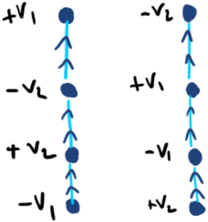
Covering Graphs

Given an oriented signed graph $\Sigma(V, E, \tau)$, the covering graph $\overline{\Sigma}(V', E', \tau')$ is created as follows:

1. For each $v \in V$, $+v, -v \in V'$
2. If $v_i v_j \in E$ is a positive edge, then $+v_i + v_j, -v_i - v_j \in E'$.
3. If $v_i v_j \in E$ is a negative edge, then $+v_i - v_j, -v_i + v_j \in E'$.
4. The orientation at $+v_i$ is "the same" as v_i.
5. The orientation at $-v_i$ is "the opposite" as v_i

Covering Graph Example

Possible B-symmetric Linear extensions of lift



Covering Relation to Root System

We can lift a signed poset on n vertices $P^{\pm}$ to an unsigned poset P on $2n$ vertices by the following:

1. If $i \neq j$, $(\epsilon_i e_i - \epsilon_j e_j) \in P^{\pm}$ iff $(e_{\epsilon_i i} - e_{\epsilon_j j}) \in P$ and $(e_{-\epsilon_j j} - e_{-\epsilon_i i}) \in P$
2. if $e_i \in P^{\pm}$ then $e_i - e_{-i} \in P$
3. if$-e_i \in P^{\pm}$ then $e_{-i} - e_i \in P$

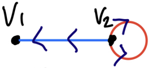
where $e_{\pm 1}, \ldots, e_{\pm n}$ form an orthonormal basis of $\mathbb{R}^{2n}$.

Example of Lifting Signed Poset

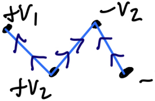
$$P^{\pm} = \{e_1 - e_2, -e_2\}$$

$$P = \{e_1 - e_2,\ e_{-2} - e_{-1},\ e_{-2} - e_2\}$$

Motivating Example

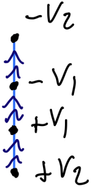
Let $\Sigma =$ 

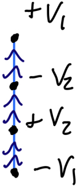
, $P^{\pm} = \{e_1 - e_2, -e_2\}$

$\bar{\Sigma} =$ 

, $P = \{e_1 - e_2, e_{-2} - e_2, e_{-2} - e_{-1}\}$

There are two β-symmetric linear extensions of $\bar{\Sigma}$

$\beta_1 =$ 

$\beta_2 =$ 

There are two elements in $\mathcal{L}(P^{\pm}) = \{\pi_1, \pi_2\}$

$\pi_1 = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}$ $\qquad$ $\pi_2 = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$

Theorem

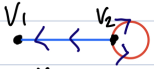
Theorem

For a signed poset $P^{\pm}$, which is associated with a signed graph Σ, every B-Sym linear extension of the covering graph of Σ can be associated with exactly one signed permutation in $\mathscr{L}(P^{\pm})$ and vice versa.

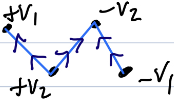
Specifically, for a linear extension, β, of the covering graph of Σ we can associate β with a signed permutation π_β which has the property that if $\beta(\epsilon v_k) =$ n-i+1, where $\epsilon \in \{+, -\}$ and n is the number of vertices, then $\pi_\beta(i) = \epsilon k$.

The theorem states that every π_β is an element of $\mathscr{L}(P^{\pm})$ and every $\pi \in \mathscr{L}(P^{\pm})$ has $\pi = \pi_\beta$ for some β which is a B-Sym linear extension of $\overline{\Sigma}$.

Motivating Example

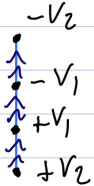
Let $\Sigma =$ 

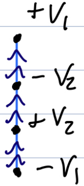
, $P^{\pm} = \{e_1 - e_2, -e_2\}$

$\overline{\Sigma} =$ 

, $P = \{e_1 - e_2, e_{-2} - e_2, e_{-2} - e_{-1}\}$

There are two β-symmetric linear extensions of $\overline{\Sigma}$

$\beta_1 =$ 

$\beta_2 =$ 

$\pi(i) = \varepsilon k$
where
$\beta(\varepsilon v_k) = n - i + 1$

There are two elements in $\mathcal{L}(P^{\pm}) = \{\pi_1, \pi_2\}$

$\pi_1 = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}$ $\quad$ $\pi_2 = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$

Intuition

It is often more useful to think about the equivalent condition
$\pi^{-1}P^{\pm} \subset \Phi^{+}$
$\pi^{-1}(k) = \epsilon i$ where $\beta(\epsilon v_k) = n - i + 1$, i.e. ϵv_k is the ith greatest element under β.

$\Phi^{+} = \{e_i\} \cup \{e_1 - e_2, e_1 - e_3, \ldots, e_1 - e_n, e_2 - e_3, e_2 - e_4, \ldots,\} \cup \{e_1 + e_2, e_1 + e_3, \ldots, e_1 + e_n, e_2 + e_3, e_2 + e_4, \ldots,\}$

If we consider Φ^{+} as a poset, then the element represented by e_1 would be maximal, the element represented by e_2 would be next maximal and so on. Therefore it makes sense to consider π such that π^{-1} sends the maximal element of P to 1 and so on.

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- The audience

References

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