Virtual Knots via the description of Gauss Codes

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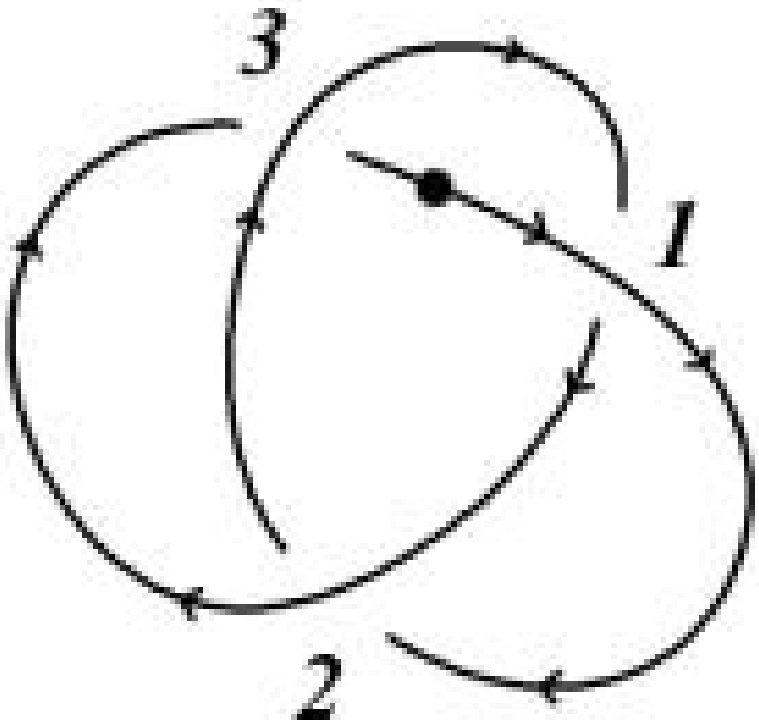
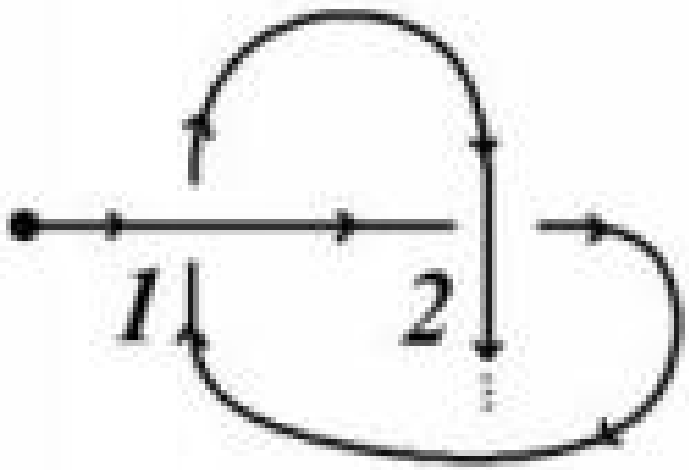
July 13, 2020

What is a Gauss Code?



O1-U2-O3-U1-O2-U3-

What *else* is a Gauss Code?



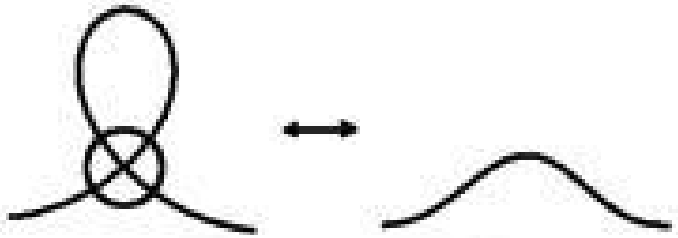
Virtual Knots in particular

The knot in the slide above can't be closed without introducing another crossing, but we can be brave and just sweep that issue under the rug by calling this new crossing "virtual" and we still have something that resembles a knot that we call a virtual knot.

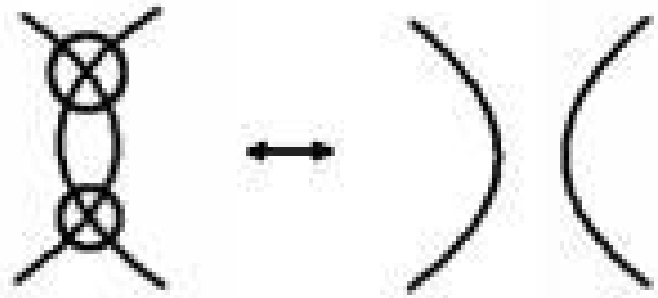
Theorem

Every Gauss Code gives rise to a (tame) virtual knot and every (tame) virtual knot diagram can be turned into a Gauss Code

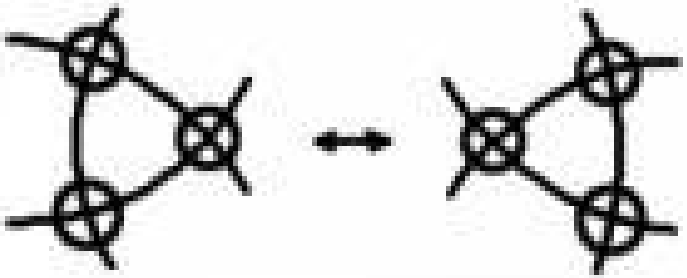
Brief review of Reidemeister moves for virtual knots



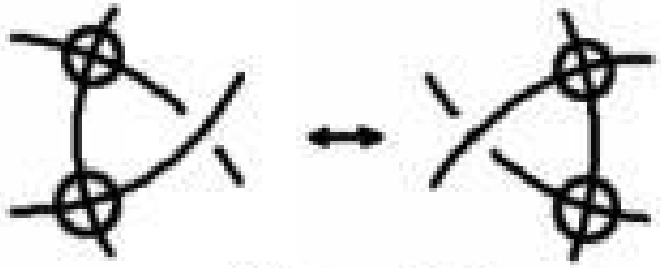
Move VI



Move VII



Move VIII



Move V

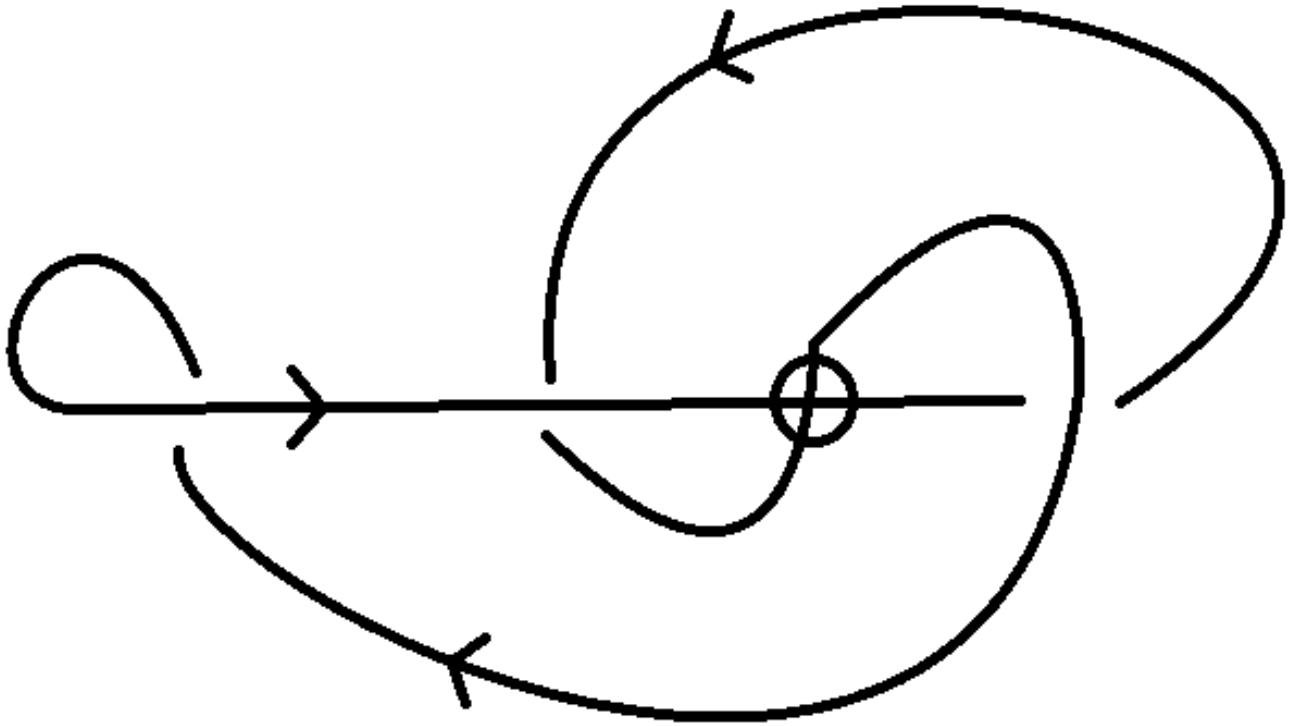
Trying an example

Example (Try to draw this Gauss Code and determine if it is a classical or virtual knot)

O1+O2-U3+U2-O3+U1+

Example illustrated in MS Paint

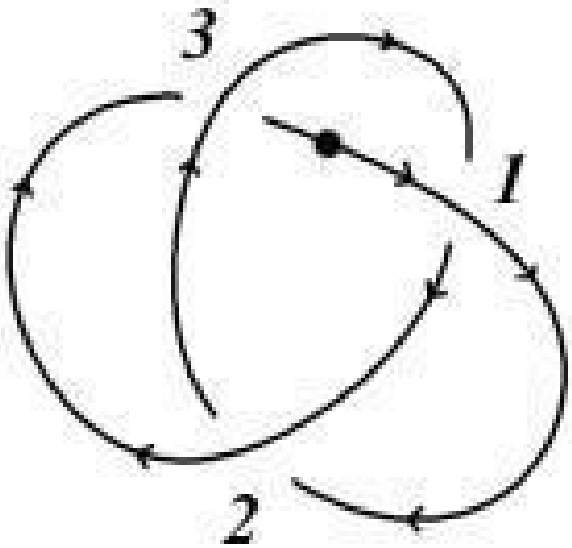
Figure: The classical crossings are 1,2,3 in order from right to left



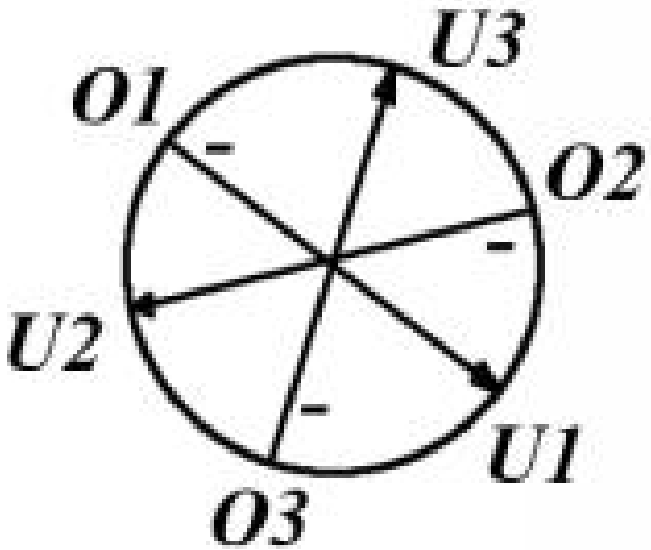
Invariance of Gauss Codes

- They aren't invariant under Reidemeister I and II because both of them reduce the number of crossings in the diagram and thus reduce the amount of terms in the Gauss Code
- Reidemeister III would permute the order of the code in a non-cyclic way (at least in the general case) and thus a code isn't necessarily invariant under it
- These facts about these codes greatly reduce the practicality of using Gauss codes for any real use other than just communicating a diagram to someone in a compact way

Gauss Diagrams come to the rescue

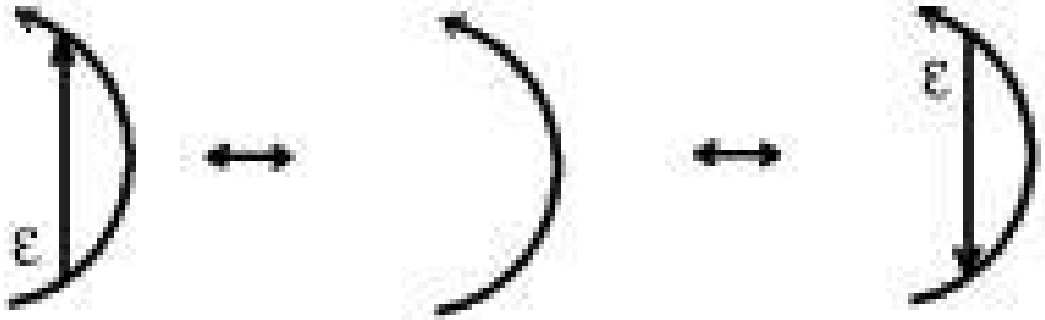


O1-U2-O3-U1-O2-U3-

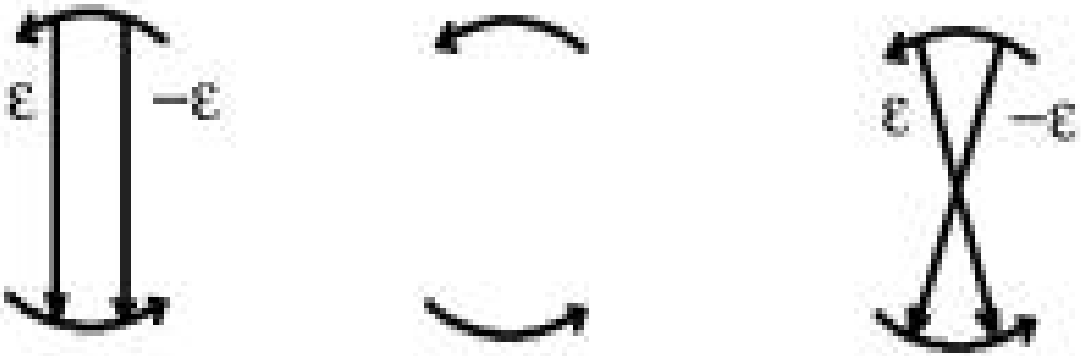


Gauss Diagram

Reidemeister moves on Gauss diagrams 1/2

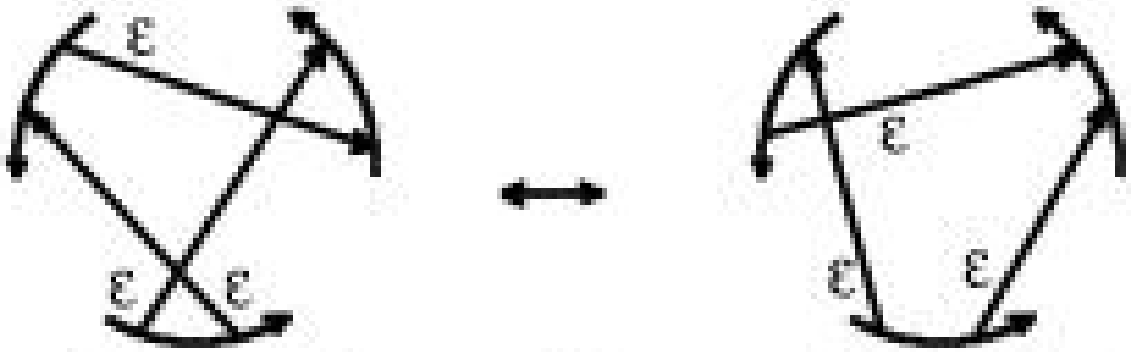


Reidemeister Type I move



Reidemeister Type II move

Reidemeister moves on Gauss diagrams 2/2

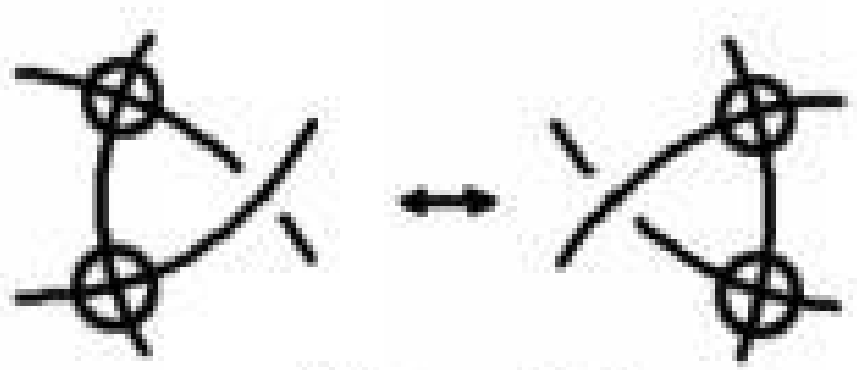


Reidemeister Type III Move

Some things about Gauss Diagrams to keep in mind

- When drawing a Gauss Diagram, virtual crossings are to be omitted
- Only the "classical" Reidemeister moves appear on the Gauss Diagram while virtual moves are taken into account as you simply add the necessary virtual crossings you need at the end
- So, when you translate a knot diagram to a Gauss diagram, use Reidemeister moves, and the translate back to knot diagram (and add the necessary virtual crossings), you have effectively performed some virtual Reidemeister moves as well

Additional interesting information 1/3



Move V

Some particularly clever people may think up of a couple more combinations of virtual Reidemeister moves that involve moving two real crossings past a virtual crossing. But, unfortunately, these moves are not correct at all (the degree to which can be seen in a bit)

Additional interesting information 2/3

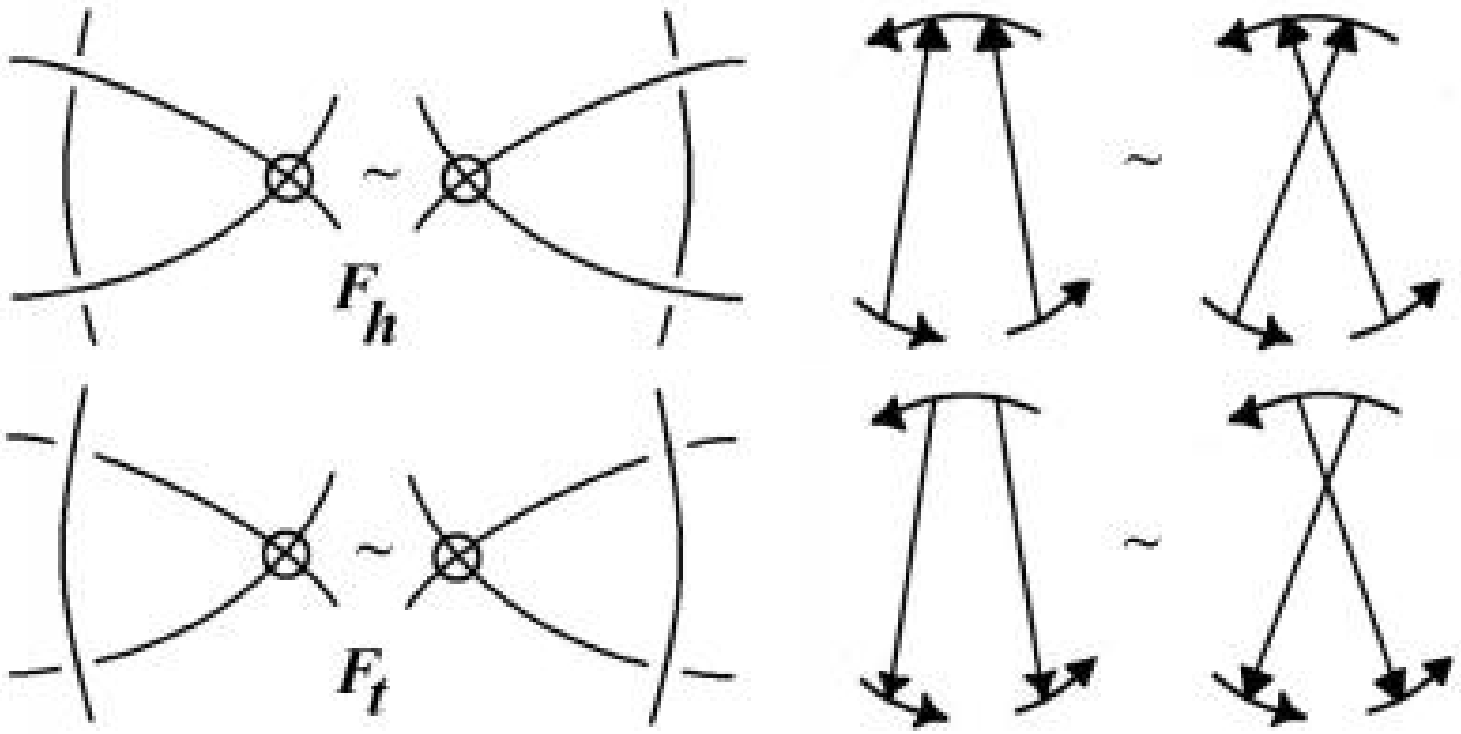


Figure: Here are the forbidden moves and their corresponding moves on the Gauss Diagram

Additional interesting information 3/3

These moves are forbidden for a very good reason:

Theorem

Any two Gauss Diagrams are equivalent under the Reidemeister moves $\cup$ *the two forbidden moves*

Proof.

See the Sam Nelson paper in the References below $\square$

Virtual Knot Group

We might be familiar with the knot group of a classical knot. But, does this make any sense with virtual knots?

Luckily, Yes

The knot group of classical knot K is $\pi_1(\mathbb{R}^3 \backslash K)$. Now, since we want virtual crossings to be "virtual", we treat the two strands of the knot at a virtual crossing to be disjoint, so the generators of the group at those two strands have no relations between them.

Definition of Virtual Knot Groups

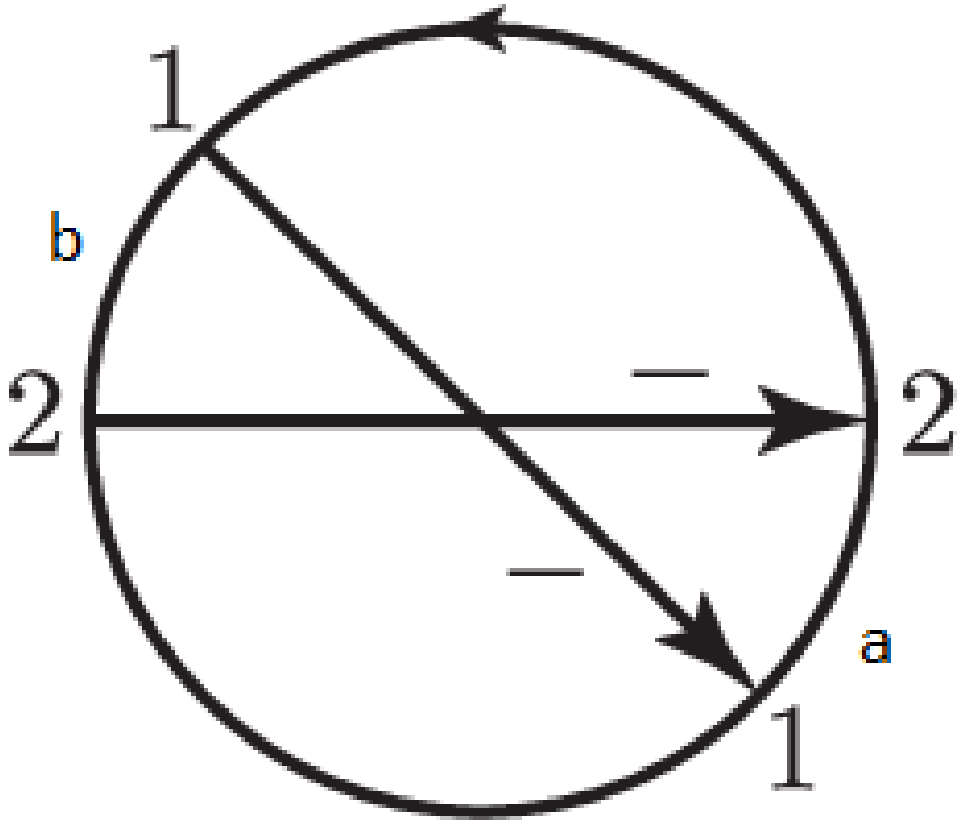
A virtual knot group can be defined nicely by using generators and relations. You add one generator for each strand between one classical undercrossing to another classical undercrossing and the relations are defined by the good old Wirtinger presentation relations. Notice, that we ignore virtual crossings all together when computing the knot group. For more information about this you can see the Louis H. Kauffman paper that first introduced Virtual knots and any source on Wirtinger presentation.

Simplified Definition of Virtual Knot Groups

Starting with a Gauss diagram, we can easily read off the Wirtinger presentation of the knot group. We have one generator for each arc in between arrow heads and there is a relation for each arrow. The procedure for reading off relations from each arrow is as follows:

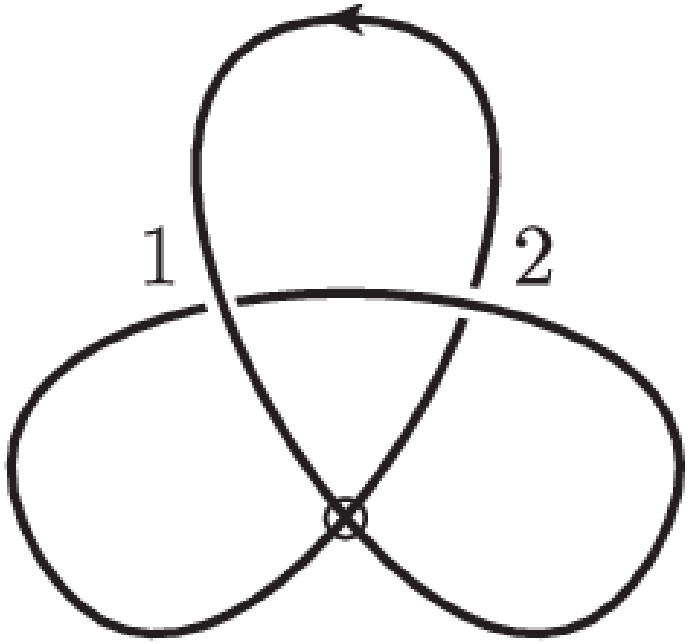
- Let ε is the sign of the given arrow
- Let a be the generator of the arc where arrow tail lies
- Let b be the generator of the arc that ends at the arrow head
- Let c be the generator of the arc that begins at the arrow head
- Then, the relation corresponding to that arrow is $c = a^{-\varepsilon} b a^{\varepsilon}$

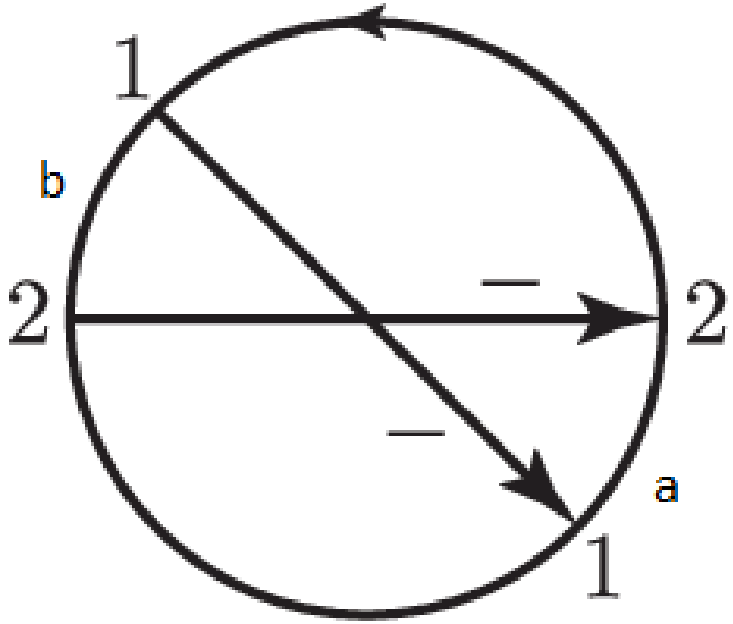
Example - Easy



What is the (virtual) knot and (virtual) knot group of this diagram?

Example - Easy (*Solved*)



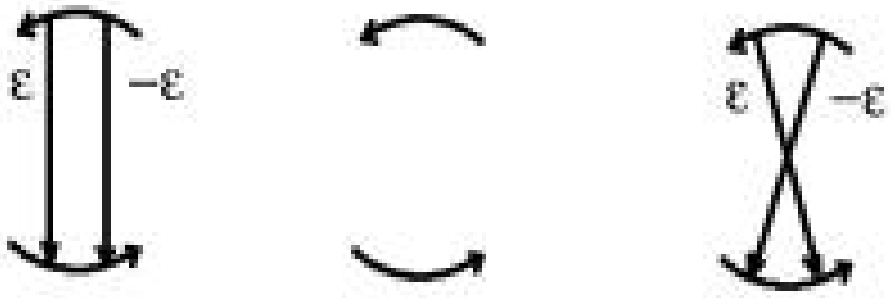


Knot Group:

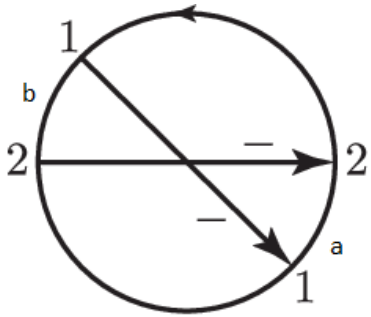
$$G = \langle a, b \mid a = b, b = bab^{-1} \rangle = \mathbb{Z}$$

Example - Easy (*Solved Easily*)

Reidemeister Type I move

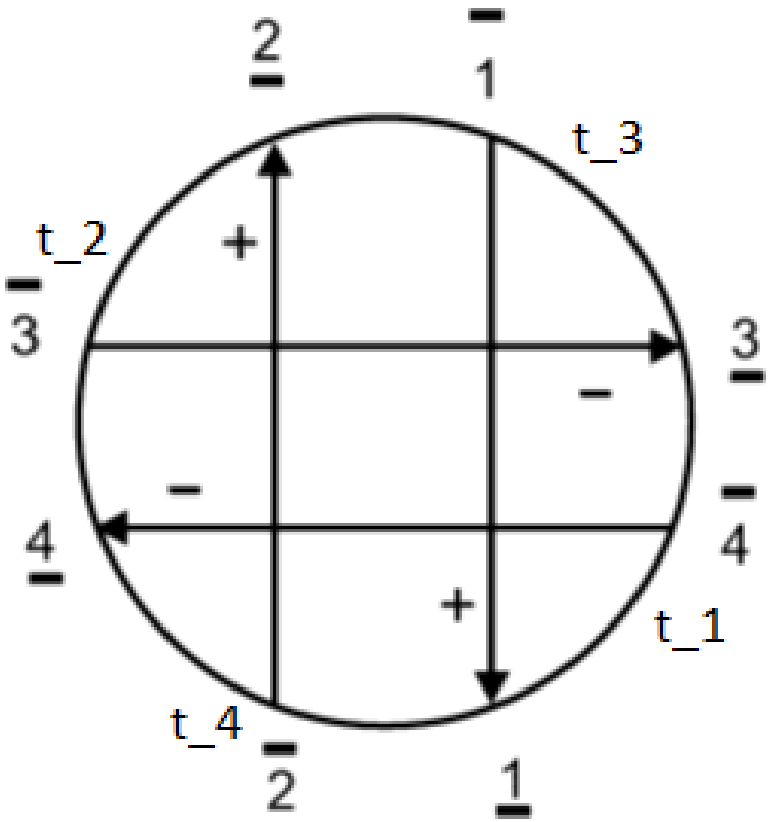


Reidemeister Type II move



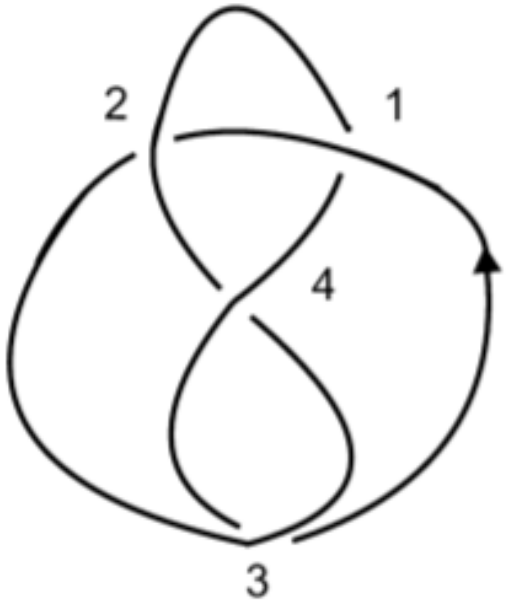
Knot Group is the same as the group of the unknot, so $\pi_1(\mathbb{R}^3 \backslash S^1) = \mathbb{Z}$

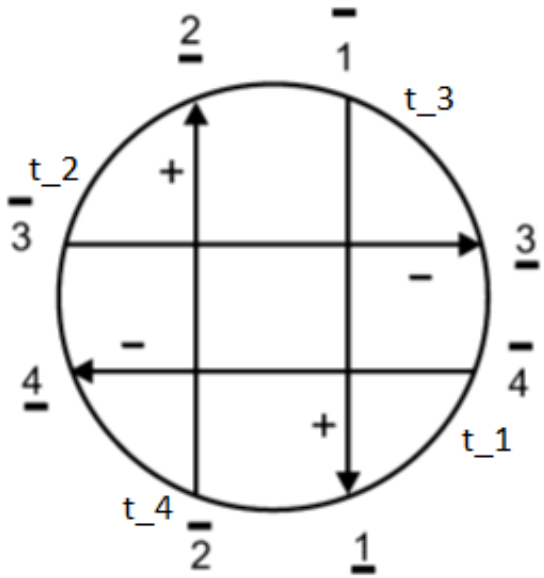
Example - Medium



What is the (virtual) knot and (virtual) knot group of this diagram?

Example - Medium (*Solved*)





Knot Group:

$$G = \left\langle t_1, t_2, t_3, t_4 \mid t_1^{-1} t_4^{t_3}, t_2^{-1} t_3^{t_4}, t_3^{-1} t_1^{-t_2}, t_4^{-1} t_2^{-t_1} \right\rangle$$

Note that: $(t_i^{t_j} = t_j^{-1} t_i t_j, t_i^{-t_j} = t_j t_i t_j^{-1})$

Example - Har... I mean, References

I have stolen most of the figures and notation from the following page introducing virtual knots by Sam Nelson:

`https://www1.cmc.edu/pages/faculty/VNelson/vknots.html`

Other than that, I used the following papers as references:

- L. H. Kauffman, Virtual knot theory. *European J. Combin.* 20 (1999) 663-690.
- M. Goussarv, M. Polyak and O. Viro. Finite type invariants of classical and virtual knots. *Topology* 39 (2000) 1045-1068.
- S. Nelson: Unknotting virtual knots with Gauss diagram forbidden moves, *J. Knot Theory Ramifications* 10 (2001), 931–935.
- S. Kim, Virtual knot groups and their peripheral structure, *J. Knot Theory Ramifications* 9 (2000), 797–812.
- K. Kaur, S. Kamada, A. Kawauchi, M. Prabhakar, Gauss diagrams, unknotting numbers and trivializing numbers of spatial graphs, *Topology Appl.*, 230 (2017), 586-589.

Bonus Section

So, given a group G with a presentation, $\langle t_1, t_2, \ldots, t_n \mid r_1, r_2, \ldots, r_m \rangle$, we can define some useful things.

Definition

The **deficiency** of a presentation is $n - m$ (the number of generator minus the number of relations).

Definition

A presentation is called a **Wirtinger** presentation if the relations are of the form $t_i^{-1} t_j^{w_j}$, where w_j are just some arbitrary words.

Definition

A presentation is called a **realizable** Wirtinger presentation if it has deficiency 0, if the relations are of the form $t_{i+1}^{-1} t_i^{\pm t_j}$ for $i = 1, 2, \ldots, n$ (there are as many relations as generators).

Bonus Section Cont.

Theorem

Any virtual knot has a knot group, and in fact, that knot group has a Wirtinger presentation of deficiency 0 or 1.

Lemma

Any Wirtinger presentation of deficiency 0 or 1 can be transformed into a realizable Wirtinger presentation.

Theorem

Any Wirtinger presentation of deficiency 0 or 1 can be realized as a virtual knot group.

Proof.

See Lemma 2 and Theorem 3 in the Kim paper mentioned in the references. □

Conclusion

The exact construction of the Gauss diagram corresponding to a given realizable Wirtinger presentation is kind of in depth and best understood by taking a piece of paper and working out a case. So, as usual, see the Kim paper for more info.