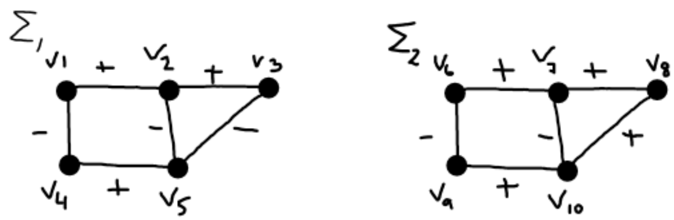
Σ = (G, σ) = (V, E, σ) - signed graph
V - set of vertices
n - # of vertices
$|\Sigma|$ = G - unsigned graph corresponding to Σ
E - set of edges
σ - sign function which labels each edge positive or negative

Balance:
An arc set S is *balanced* if each circular path in it has positive product of signs
Example:



Let $S_1 = E(\Sigma_1),\ S_2 = E(\Sigma_2)$, and $S_3 = \{v_6v_7, v_7v_{10}, v_{10}v_9, v_9v_6\}$

Positive Cycles:
$v_1v_2v_5v_4v_1$
$v_2v_3v_5v_2$
$v_1v_2v_3v_5v_4v_1$
$v_6v_7v_{10}v_9v_6$

So S_1 is balanced and S_2 is not.

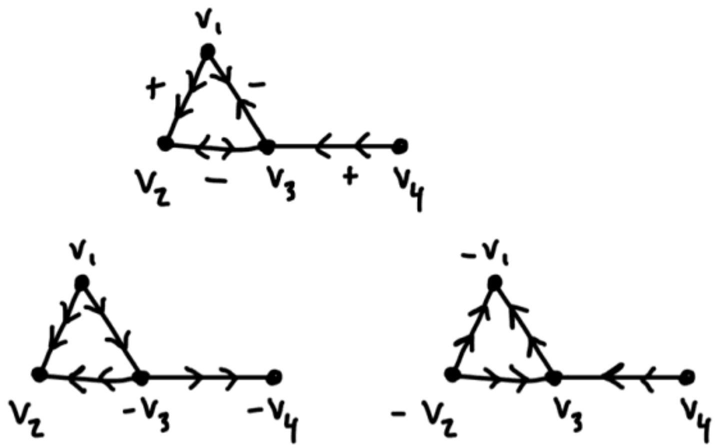
Def: $\pi_b(S) = \{W\text{: } W \text{ is the node set of a balanced components of } S \text{ and } W \neq \emptyset\}$

$\pi_b(S_1) = \{\{v_1, v_2, v_3, v_4, v_5\}\}$
$\pi_b(S_2) = \emptyset$
$\pi_b(S_3) = \{\{v_6, v_7, v_9, v_{10}\}, \{v_8\}\}$

Coloring:
$k\colon N \rightarrow \{-\lambda, -\lambda+1, \ldots, 0, \ldots \lambda-1, \lambda\}$
Zero-free coloring never assumes the value of zero.
Improper coloring: given edge $e\colon vw$, and coloring k, e is improper if
$k(w) = \sigma(e)k(v)$
The set of improper edges of coloring k, denoted $I(k)$, may also be called the set of impropriety of k

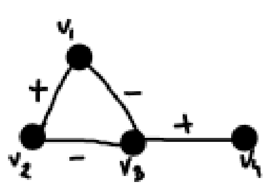
Lemma: The set of impropriety of a coloring k is balanced if k is zero-free.

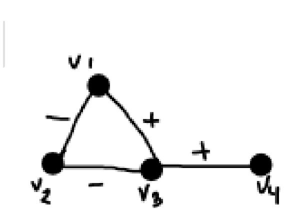
Signed Covering Graph $\tilde{\Sigma}$:
$N(\tilde{\Sigma}) = \pm N(\Sigma)$
e: vw with sign ε is covered by two arcs: $\tilde{e}_1$: $+v, \varepsilon w$ and $\tilde{e}_2$: $-v, -\varepsilon w$
Covering projection:
p: $\tilde{\Sigma} \to \Sigma$, where $p(\pm v) = v$ and $p(\tilde{e}) = e$
Covering orientation:
$\tilde{\tau}(\varepsilon v, \tilde{e}) = \varepsilon\tau(v, p(\tilde{e}))$
$\tilde{k}(\varepsilon v) = \varepsilon k(v)$



Notes:
τ is acyclis iff $\tilde{\tau}$ is
$\tilde{\Sigma}$ is an unsigned graph

Switching:
υ : V → {±1}
Switching by υ means reversing the sign of any edge whose endpoints have opposite υ value.
Switching a vertex switches its color to the opposite.





Exercise: show that switching does not alter balance and preserves sets of impropriety.

Deletion and Contraction:

Deletion works the same as with unsigned graph.

To contract an edge e:

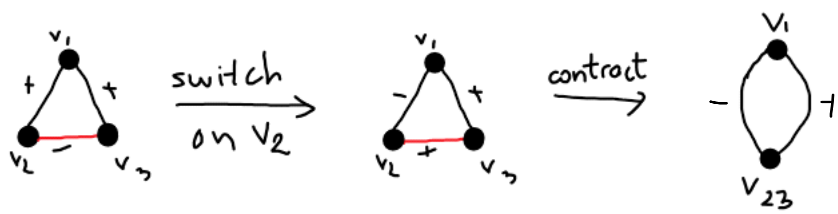
If e is a positive edge, same process as with unsigned graph.

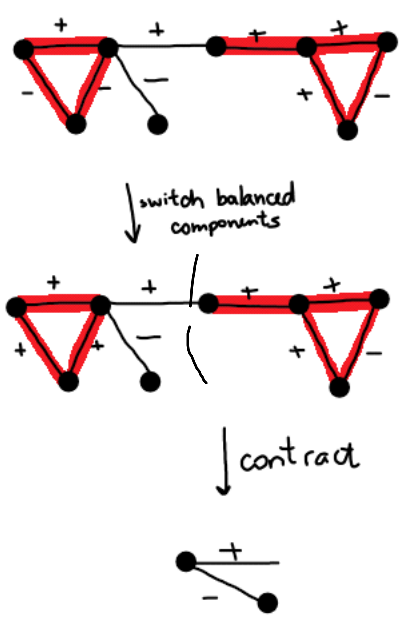
If e is a negative edge, switch Σ such that e becomes a positive edge. Then contract e as a positive edge.

To contract an edge set S:

$E(\Sigma/S) := E \setminus S,$ $\quad V(\Sigma/S) := \pi_b(\Sigma|S) = \pi_b(S)$

Switch Σ so every balanced component of S is all positive, coalescing all the nodes of each balanced component, and discarding the remaining nodes and all the edges in S.





The importance of contraction come when we contract the graph on its set of improprieties of a coloring.

Lemma 1.5. *Let Σ be a signed graph and $\mu \geqslant 0$ an integer. There is a one-to-one correspondence between all signed colorings of Σ in μ colors and all proper signed colorings of contractions of Σ, in which the zero-free signed colorings correspond to the zero-free proper colorings of contractions by balanced sets.*

A coloring k of Σ corresponds to the proper coloring k' of $\Sigma/I(k)$ determined by first switching Σ (and k) until every balanced component of $I(k)$ is positive, then defining $k'(B) = k(v)$ for $v \in B \in \pi_b(I(k))$.

A proper coloring k' of Σ/A corresponds to the coloring k of Σ determined by first switching Σ until every balanced component of A is positive, then defining $k \mid B = k'(B)$ for each $B \in \pi_b(A)$ and $k \mid N_u(A) = 0$, then reversing the switching of Σ and k.