Thompson's Group and Regular Isotopy of Links

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Thompson's Group F

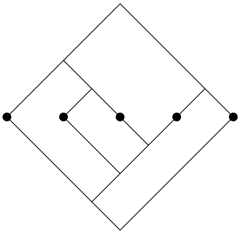
Let F be the set of piecewise-linear continuous bijections $f : [0, 1] \rightarrow [0, 1]$ satisfying

- If f is differentiable at x, then $f'(x)$ is a power of 2
- $f(0) = 0$
- There are finitely many points at which f is not differentiable, and they are all *dyadic rationals*, i.e., rational numbers equal to $\frac{a}{2^b}$ for some integers a, b.

F forms a group under composition.

Binary Trees

Every element of F can be represented by a pair of binary trees with the same number of leaves.



Here, $[0,1/4],[1/4,3/8],[3/8,1/2],[1/2,3/4]$, $[3/4,1]$ map linearly onto $[0,1/4],[1/4,3/8],[3/8,7/16],[7/16,1/2]$, $[1/2,1]$, respectively.

For our talk, we will identify an element of F with its (unique) depiction as a pair of binary trees with a minimal number of leaves.

A Link?

A *link* is an embedding of finitely many disjoint circles in $\mathbb{R}^3$.

Two links are considered "equivalent" if one can be continuously deformed to the other in $\mathbb{R}^3$.

Formally, links $L_1, L_2 \subseteq \mathbb{R}^3$ are "equivalent" if there is an orientation-preserving homeomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(L_1) = L_2$.

So that we can draw them, links are usually depicted by "link diagrams".

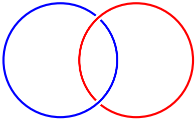


Figure: A link diagram for the Hopf Link.

Link diagrams are "equivalent" if they depict equivalent links.

Creating links from elements of F

In 2014, Vaughan Jones developed a method to produce unitary representations of F.

In some of these representations, links arose as coefficients, so that for each $g \in F$, one can associate a link diagram $L(g)$. [4].

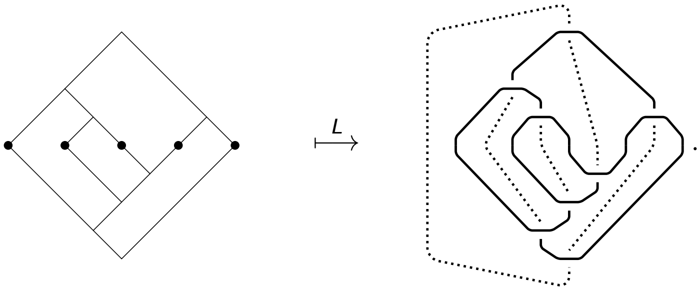


Figure: An example of the mapping L.

Which link diagrams are produced by L?

Theorem (Jones, 2014)

For each link diagram D, there is a $g \in F$ such that $L(g)$ is equivalent to D.

In some of Jones's representations, classes of link diagrams subject to other equivalence relations arise.

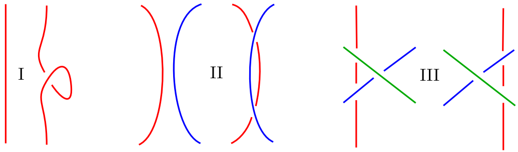
One equivalence relation on link diagrams, which is finer than link diagram equivalence, is called "regular isotopy" of link diagrams.

In 2018, Jones asked if for every link diagram D, there is a $g \in F$ such that D is regular isotopic to $L(g)$. [5]

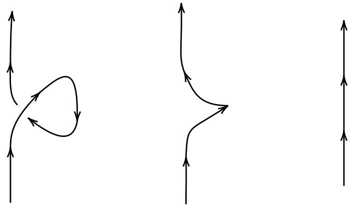
We answer this question in the negative, and classify the diagrams D for which this is true.

Motivating Regular Isotopy

- knots and links = diagrams up to Reidemeister moves:

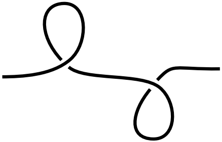


- One of these moves is not like the others.
- Move I can't keep the curve smooth in the plane.



Regular Isotopy

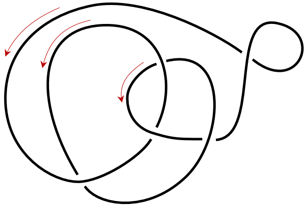
- Link diagrams A and B are said to be *regular-isotopic* if A can be transformed into B using only moves II and III.
- Excercise: Show that the following two diagrams are regular-isotopic:



- Refining Jones's result: which regular isotopy classes come from Thompson's group?
- No cusps $\Longrightarrow$ more invariants for regular isotopy?

Invariant 1: Whitney Index

- The *Whitney index* of a smooth oriented knot diagram is its total curvature divided by 2π, i.e. the winding number of the derivative vector.
- The following curve has Whitney index $+2$.

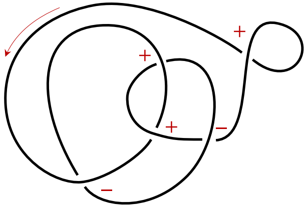


- See [7] for a sum-over-crossings formula.
- Exercise: verify that Whitney index is invariant under regular isotopy, but not Reidemeister move I

Invariant 2: Writhe

- The *writhe* of a smooth oriented knot diagram is the sum of the *signs* of the crossings, assigned as follows:

- The following curve has writhe $+1$:



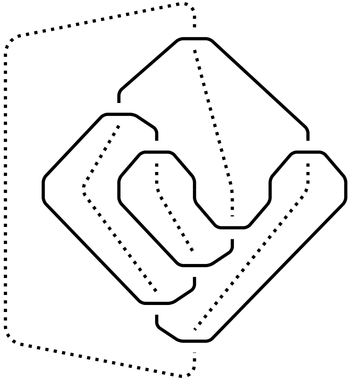
- Invariant under regular isotopy, but not Reidemeister move I

Writhe and Whitney index are all we need

- By a theorem of Alexander Coward [3], if two diagrams of the same link have the same writhe and Whitney index for each corresponding component, then the diagrams are regular-isotopic.
- Goal: describe regular isotopy classes attainable from Thompson's group based on their writhe and Whitney index.

Restrictions for Regular Isotopy Classes from F

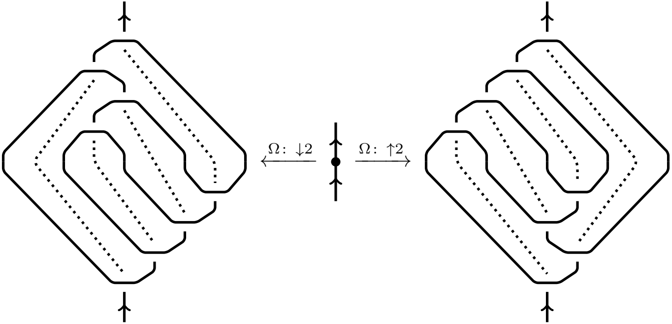
- A parity restriction: for any link coming from F and each component C of that link, each crossing on top where C is the under-strand corresponds to exactly one such crossing on bottom.



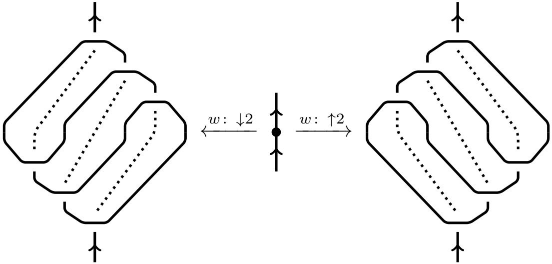
Restrictions for Regular Isotopy Classes from F

- Each component is the understrand of an even number of crossings.
- For a given link, this fixes the parity of number of self-crossings that each component has.
- Parity of number of self-crossings determines parity of writhe and Whitney index.
- This necessary condition is also sufficient:
- Assuming your components each have an even number of under-crossings, we can produce an element of F that lands in the same regular isotopy class. Proof:
  1. Generate an element of F that gives the right link (Jones's construction).
  2. Adjust its writhe and Whitney index by even numbers.

Adjusting the Whitney index of a component



Adjusting the writhe of a component



This completes the proof.

Loose end: The Oriented Case

- Some elements of F admit a canonical orientation on each component. These elements form a subgroup known as $\vec{F}$. See [4].
- The condition on regular isotopy classes to be represented by $\vec{F}$ works over $\mathbb{Z}$ rather than $\mathbb{Z}/2\mathbb{Z}$: the linking matrix must have columns sum to zero.
- The moves required are slightly more complicated, but the same structure of the proof applies.

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