Thompson's Groups

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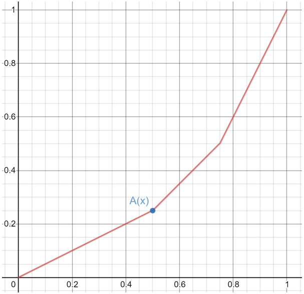
Some Special Functions on $[0, 1]$

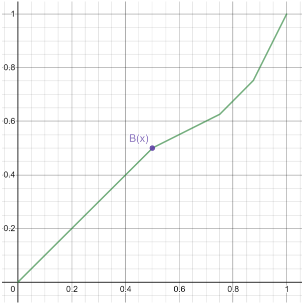
Let F be the set of piecewise-linear continuous bijections $f : [0, 1] \to [0, 1]$ satisfying

- If f is differentiable at x, then $f'(x)$ is a power of 2
- There are finitely many points at which f is not differentiable, and they are all *dyadic rationals*, i.e., rational numbers equal to $\frac{a}{2^b}$ for some integers a, b.

Examples

$$A(x) = \begin{cases} \frac{x}{2}, & 0 \le x \le \frac{1}{2} \\ x - \frac{1}{4}, & \frac{1}{2} \le x \le \frac{3}{4} \\ 2x - 1, & \frac{3}{4} \le x \le 1 \end{cases} \quad B(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4}, & \frac{1}{2} \le x \le \frac{3}{4} \\ x - \frac{1}{8}, & \frac{3}{4} \le x \le \frac{7}{8} \\ 2x - 1, & \frac{7}{8} \le x \le 1 \end{cases}$$





A group?

Recall that a group is a set G, along with a binary operation, satisfying the following properties:

- There is an element $e \in G$ such that for all $g \in G$, $ge = eg = g$,
- For all $g \in G$, there is an element $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$,
- For all $g, h, k \in G$, $g(hk) = (gh)k$.

Let's check that F, under the operation of composition, is a group.

A group! Part 1: Inverses

Let $f \in F$, and let $0 = a_1 < a_2 < \cdots < a_n = 1$ be the points at which f is not differentiable.

Since f is increasing, $0 = f(a_1) < f(a_2) < \cdots < f(a_n) = 1$. For $i < n$ and $x \in [a_i, a_{i+1}]$, there is an integer k_i such that $f(x) = 2^{k_i}(x - a_i) + f(a_i)$. By induction on n, $f(a_i)$ is a dyadic rational for each i, and therefore f sends dyadic rationals to dyadic rationals.

For $x \in [f(a_i), f(a_{i+1})]$, $f^{-1}(x) = 2^{-k_i}(x - f(a_i)) + a_i$. So, $f^{-1} \in F$.

A group! Part B: Composition

Let $f, g \in F$. Clearly, $f \circ g$ is a continuous bijection from $[0, 1]$ to $[0, 1]$.

If $f \circ g$ is not differentiable at x, then either g is not differentiable at x, or f is not differentiable at $g(x)$. Thus, $f \circ g$ is not differentiable at finitely many points.
If g is not differentiable at x, then x is a dyadic rational. If f is not differentiable at $g(x)$, then $g(x)$ is a dyadic rational, so x is a dyadic rational.

Exercise: Show that if $f \circ g$ is differentiable at x, then the derivative at x is a power of 2. This is easy if g is differentiable at x and f is differentiable at $g(x)$, but a subtlety arises if this is not the case.

Why do we care?

F was first proposed in the 1960s as a potential counterexample to the famous *von Neumann conjecture*, about the *amenability* of certain finitely presented groups. It is still not known if F is amenable, but many are interested in this question.

F has been repeatedly discovered by topologists as an example arriving in surprising settings.

F, and its related supergroups T and V have many surprising properties that make them objects of special interest. Many are quite technical, but one that you may already find surprising is that F has the finite presentation $\langle A, B : [AB^{-1}, A^{-1}BA], [AB^{-1}, A^{-2}BA^2] \rangle$.

F has a nice geometric interpretation, which coupled with its vast array of interesting properties, makes it an important object of study in geometric group theory.

Why do *we* care?

In 2018, Vaughan Jones developed a way to produce links from elements of Thompson's group.

In 2019, Valeriano Aiello showed that any link can be produced in this way.

Thus, just as a link diagram can represent a link, a given link can be represented by an element of Thompson's group.

Our project is an attempt to find the corresponding "Reidemeister moves" for elements of Thompson's group; in other words, we wish to determine exactly when two elements of F produce the same link.

References

J.W. Cannon, W. J. Floyd, W. R. Perry: Introductory Notes on Richard Thompson's Groups

V. Jones: On the Construction of Knots and Links from Thompson's Groups

Wikipedia