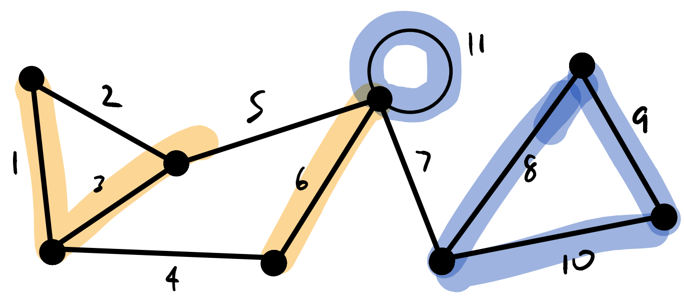
What IS a Matroid?

A matroid is a "ground set" E with a set of "independent" subsets $I \subseteq 2^E$ that satisfy some "nice" criteria.

Let's Look at Examples first.

Definition: On a graph, call a set of edges independent if it contains no cycles.



What are some independent sets in the graph above?

$\{1\}$, $\{6\}$, $\{8\}$, $\{1, 3, 6\}$

$\{1, 2, 5, 6, 7, 10, 9\}$

Dependent Set $\{8, 9, 10\}$

$\{11\}$

$\{5, 12\}$

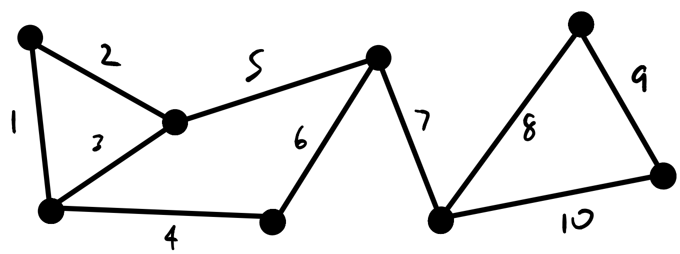
$$G = (V, E) \longrightarrow M = (E, \mathcal{I})$$

What can we say about $\mathcal{I} = \{ A \subseteq E \mid A \text{ is cycle-free} \}$?

1) $\emptyset$ is always cycle-free

2) Any subset of a cycle-free set is cycle-free.

This is good and all, but cycles have a little more structure that can be captured.



What other properties does $\mathcal{I}$ have?

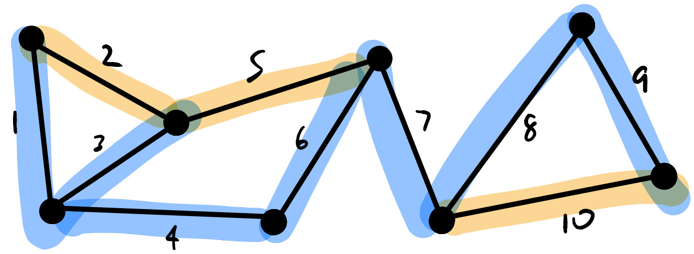
1) Closed under intersection

2) Spanning trees are maximal, all independent subsets are subsets of trees

3) $\mathcal{I}/e = \{ A \subseteq E \setminus e \mid A \cup \{e\} \in \mathcal{I} \}$

4) 7 can be added any independent set

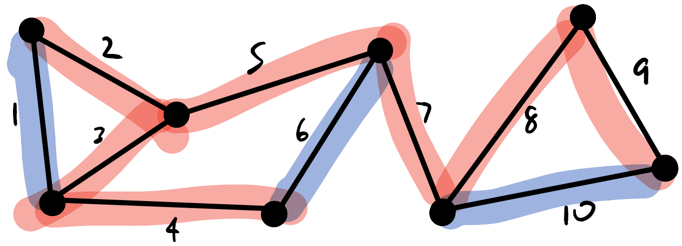
T_1

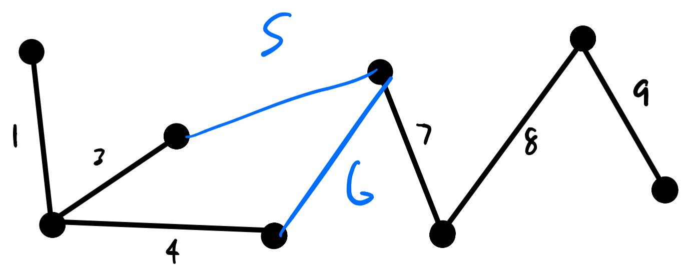


$6 \in T_1$

$6 \notin T_2$

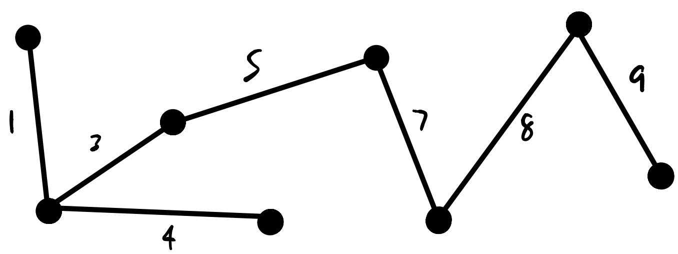
T_2





$5 \in T_2$

$5 \notin T_1$



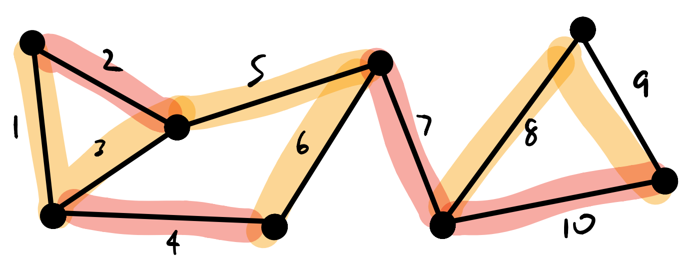
Definition of Matroid (Basis Edition)

A matroid $M = (E, B)$ is a set of "edges" E with a set B of "bases" $T \subseteq E$ such that...

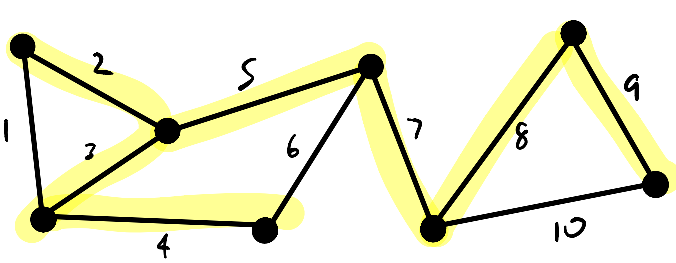
1) B is nonempty
2) If $T_1, T_2 \in B$, then for any $a \in T_1 \setminus T_2$, there exists $b \in T_2 \setminus T_1$ such that $(T_1 \setminus a) \cup b \in B$

If you have two trees T_1 and T_2, you can replace any differing edge in T_1 with an edge in T_2.

B



A



$|A| > |B| \geq$ # of edges that create cycle

Definition of <u>Matroid</u> (Independence Edition)

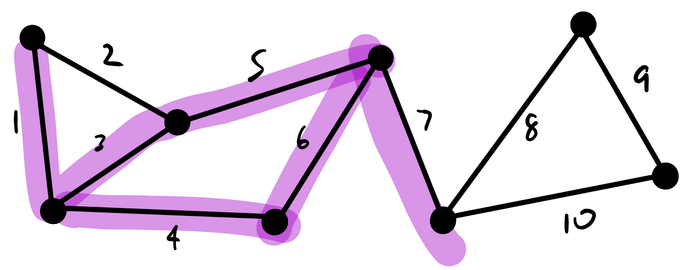
A matroid $M = (E, I)$ is a set of edges E with a set of Independent subsets I such that...

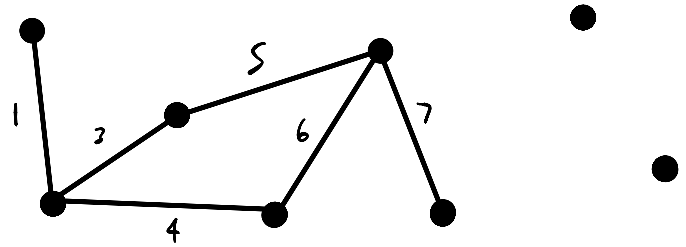
1) $\phi \in I$
2) Subsets of independent sets are independent
3) If A and B are independent and $|B| < |A|$, then there is some $x \in A \setminus B$ such that $B \cup \{x\} \in I$.

<u>Trees (aka bases) are maximal independent sets.</u>

<u>Cycles are minimal dependent sets.</u>

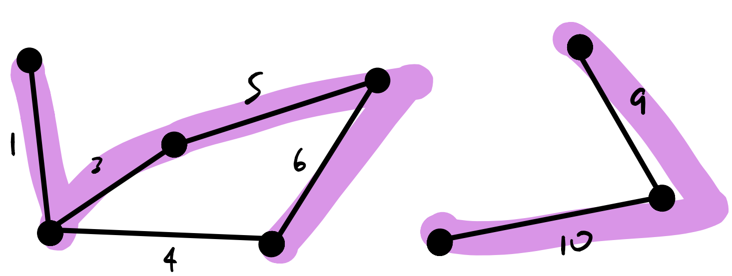
If $S \subseteq E$, then we can consider the matroid restricted to S, $M|_S$.





Definition of <u>Rank</u> of a subset $S \subseteq E$

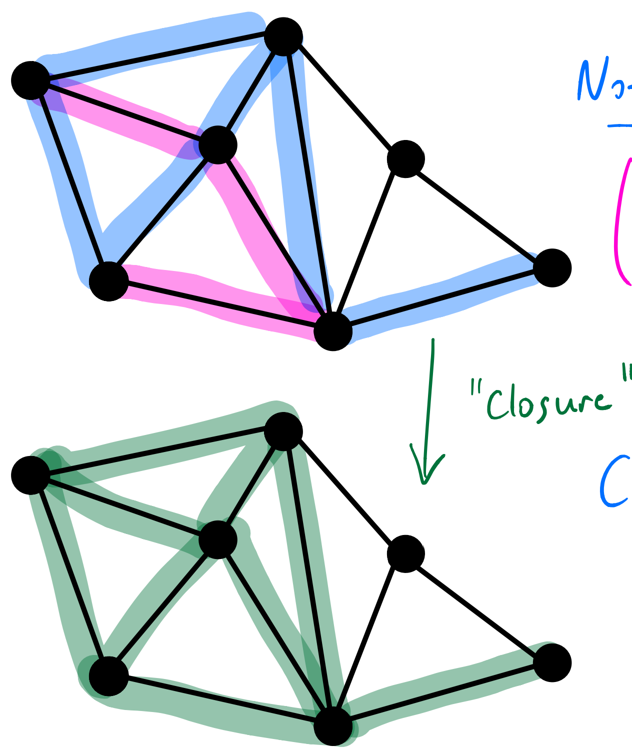
$r(S)$ is the size of the bases in $M|_S$, i.e. The number of edges in the maximal spanning forests of the restriction to S.



$r(S) = 6$

Definition of <u>Flats</u> of a Matroid.

A set V is a flat of M if adding any edge to V would increase its rank.



Not a flat

(pink edges could be added without increasing the rank)

Closure $cl(S)$ gives a flat with

$$r(cl(S)) = r(S)$$

The Tutte Polynomial

Can be expressed on matroids.

For graphs G,

$$T_G(x,0) \propto \chi_G(1-x) \quad \text{(chromatic)}$$

$$T_G(0,y) \propto F_G(1-y) \quad \text{(flow)}$$

For matroids M,

$$T_M(x,y) = T_{M^*}(y,x).$$

$$T_M(x,y) = \sum_{A \subseteq M} T_{M|_A}(0,y)\, T_{M/A}(x,0)$$

$$\sum \sim \chi_{M/A}(1-x) \sim \chi_{\cdots}(1-y)$$

Situation

Goal A symmetric Tutte Polynomial

What we have

1) A good symmetric chromatic polynomial
(The Stanley Symmetric Chromatic Function)

2) A good way of dealing with contraction
and homogeneous degrees (Sesquigraphs)

3) A way of simplifying the Tutte Polynomial
into a product of chromatic polynomials

What's Next?

Jam the Stanley Chromatic Function into the Convolutional Formula!!

$$C_G(\bar{p}, \bar{q}) \equiv$$

$$\equiv \sum_{S \subseteq E} (-1)^{|A/S| + |B/(E-S)|} X_{A/S}(\bar{p}) X_{B/(E-S)}(\bar{q})$$

It turns out that

$$C_G(-\bar{p}, -\bar{q}) = D_G(\bar{p}, \bar{q})$$