Tutte Talk

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Introduction

The Tutte polynomial is defined in many ways, for one as

$$\sum_{F\subseteq E}(x-1)^{k(F)-k(E)}(y-1)^{k(F)+|F|-|V|}$$

but, with a bit of work, another definition, by activity, is much better for seeing the properties of the polynomial.

Cut(T,e) and Cyc(T,e)

Let T be a spanning tree of G. If $e \notin T$, then $T \cup e$ contains a cycle, as T is connected and hits every vertex, so there is a path from one end of e to the other within T.

Definition 1. *We denote the cycle formed by $T \cup e$ as Cyc(T,e)*

Notice that $e \in Cyc(T,e)$, and that $Cyc(T,e)$ is usually strict subset of $T \cup e$. Now, suppose $e \in T$. We first define Cut(T,e) by defining what a cut is.

Definition 2. *$X \subset G$ is a cut if there exists a partition $V = V_1 \cup V_2$, so that every edge $e = uv$ in X has $u \in V_1$ and $v \in V_2$ or vice versa.*

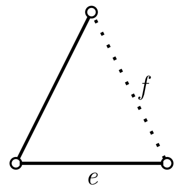
We then define an induced cut of a partition as the largest set X that is a cut over the given partition. Now, we see that if $e \in T$, then $T - e$ is a disconnected subgraph. So, $T - e = A \cup B$.

Now, we can define $V_A = \{v \in V | \exists e \in T - e v \text{ adjacent to e}\}$, and V_B similarly. However, every vertex is still adjacent to an edge in $T - e$, so V_A and V_B partition V.

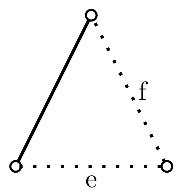
Definition 3. *If $e \in T$ Cut(T,e) is the induced partition of V_A and V_B*

While this definition is a little obtuse, one can think of it simply as the edges that "reconnect" the spanning tree. Let's work out an example on the complete graph of 3 vertices.

Example 1. *If G is a triangle, then T =*



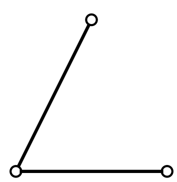
Cyc(T, f) would simply be the entire graph, while cut(T, e) is the set of points "reconnecting" T - e:



Which can be e or f in this case.

Cyc(T, e) and Cut$(\overline{T}, e')$ are dual notions

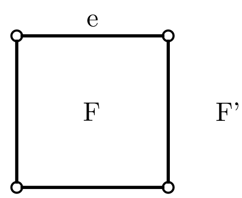
First, we need to talk about what $\overline{T}$ is. We know that there is a natural association between edges of a planar graph G and its dual, G^*. As such, for any spanning tree, we can talk about it associated graph T^*. As it turns out, though, T^* is not a tree in general (or even most of the time). However, its complement, $\overline{T} = G - T*$ For an example, take the triangle graph once again. As per usual, T =



As such, T^* is

Which makes $\overline{T}$

Hopefully you buy now that $\overline{T}$ is in fact a spanning tree of G^*. As stated above, Cyc(T, e) and Cut$(\overline{T}, e')$ are dual notions. In fact, they are equal, in the sense that the image of the edges in Cyc(T, e) form Cut$(\overline{T}, e')$. A good way to see this is by drawing a cycle, C in the original graph on top of the it image in the dual graph, $\overline{C}$:



Here, $\overline{C} = F \cup F'$ As you can see, there can be no edge connecting F and F' because if there was, then it would be associated to an edge in C, which is a contradiction as this would imply the edge is not in $\overline{C}$.

Now, if e (shown above) is not is the spanning tree, and Cyc(T,e) = C, then we see that the image of e in the dual graph (e') is in $\overline{T}$, and, the only ways to "reconnect" $\overline{T} - e'$, are through the the cycle C, so the edges are the same.

Activity and the Tutte Polynomial

Given an ordering of edges, and a spanning tree T, we say an element $e \notin T$ is *externally active* if e is the least element of $Cyc(T, e)$, and we say $e \in T$ is *internally active* if e is the least element of $Cut(T, e)$. Note that by the previous section, e is internally active iff it is externally active in the dual graph. We now know enough to define the Tutte polynomial.

Definition 4. *Given an ordering on a graph G, we define the Tutte polynomial as*

$$T_G = \sum_{T \subset G} x^{\#of\ internally\ active\ edges} y^{\#of\ externally\ active\ edges}$$

It not obvious that the Tutte polynomial is independent of ordering, but it was proven that this is the case.

Proposition 1. T_G *is independent of ordering*

Proof. The general strategy of the proof is to prove that the polynomial remains unchanged from one ordering $= \{e_1 < e_2 < \dots e_i < e_{i+1} \dots e_m\}$ to another ordering with two adjacent elements switched, $\{e_1 < e_2 < \dots e_{i+1} < e_i < \dots e_m\}$. Now, every ordering can be associated to an element of the symmetric group S_m, and changing orderings in this way is the same as composing these with adjacent switches, which as we know generate the group. □

There are many nice properties of the Tutte polynomial that fall out quickly from this definition. For one, the polynomial is obviously symmetric over the Dual graph as internally active is dual to externally active.

Theorem 2. $T_G(x, y) = T_{G^*}(y, x)$

Additionally, I claim that deletion contraction is easy to prove using this definition as well.

Theorem 3.

$$T_G(x,y) = \begin{cases} xT_{G/e}(x,y) & \textit{if e is a bridge} \\ yT_{G-e}(x,y) & \textit{if e is loop} \\ T_{G-e}(x,y) + T_{G/e}(x,y) & \textit{otherwise} \end{cases}$$

Proof. We'll do the proof for the special cases. If e is a bridge, then it's included in every spanning tree, and when you remove it from the tree it is the only element that can be added to reconnect the tree. Therefore, it must be internally active for all T, which is the same as multiplying by x.

For the case where e is a loop, you can simply apply the symmetry theorem and then the case where e' is a bridge and then go back. □