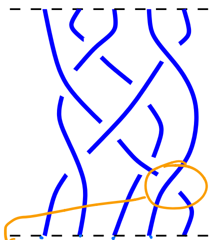
A group is a set $S = \{a_1, a_2, \dots\}$ with an operation $S \times S \to S$

1) $\exists\, 1 \in S$ $\quad 1 \cdot a = a \cdot 1 = a$

2) $\forall a \in S$ $\exists\, a^{-1} \mid a \cdot a^{-1} = a^{-1} \cdot a = 1$

3) $(ab)c = a(bc)$

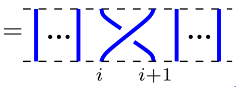
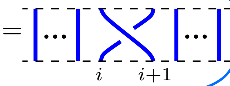
Braids



$\sigma_4\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_3\sigma_1^{-1}\sigma_4^{-1}\sigma_2$

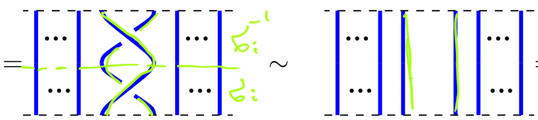
Elements of the braid group $\mathfrak{B}_n$ have topological interpretation as a collection of n strands going monotonically vertical from fixed n points at the bottom of the braid to the same points at the top of braid considered up to a homeomorphism of the slice of $\mathbb{R}^3$ between the bottom and the top.

The group operation is staking the second braid at a top of the first one. The identity element consists of n vertical strand without any crossings.

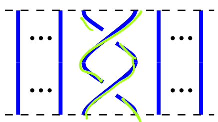
Let $\sigma_i :=$  , then $\sigma_i^{-1} :=$  because

$\pi = \begin{matrix} 1 & 2 & 3 \\ \downarrow & & \\ 3 & 2 & 1 \end{matrix}$

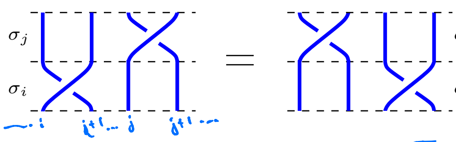
not a braid

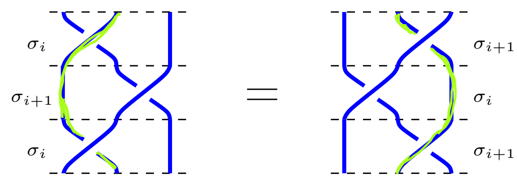
$\sigma_i\sigma_i^{-1} =$  $= 1 \in \mathfrak{B}_n$

The square $\sigma_i^2 \neq 1$:

$\sigma_i^2 =$  $\neq 1 \in \mathfrak{B}_n$

The relations in the braid group look like this.

$\sigma_i\sigma_j =$  $= \sigma_j\sigma_i$



$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

So the braid group $\mathfrak{B}_n$ has the presentation by generations and relations

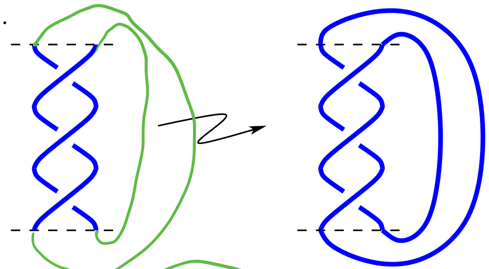
$$\mathfrak{B}_n := \langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} | \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |j-i| > 1, \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle$$

There is always an epimorphism of $\mathfrak{B}_n$ onto the symmetric group

$$S_n := \langle s_1, s_2, \dots, s_{n-1} | s_i^2, \quad s_is_j = s_js_i \text{ if } |j-i| > 1, \quad s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \rangle,$$

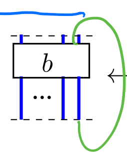
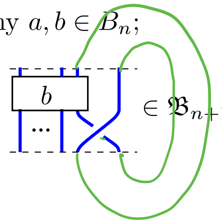
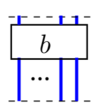
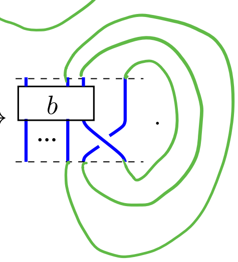
which sends a braid to a permutation of the endpoints. Algebraically this epimorphism is given by adding the relations $\sigma_i^2 = 1$ to the relation of the braid group and replacing $\sigma_i \mapsto s_i$. The kernel of this epimorphism is called *pure braid* group, $\mathfrak{P}_n$.

Theorem. Closure of braids. [Alexander, 1923] *Any link can be represented as a closure of a braid.*



Markov theorem.[Mark, Bir1] *Two closed braids are equivalent (as links) if and only if the braids are related by a finite sequence of the following Markov moves.*

? (M1) $\qquad b \longleftrightarrow aba^{-1}$ for any $a, b \in B_n$;

? (M2) $\qquad \mathfrak{B}_n \ni$  $\longleftrightarrow$  $\in \mathfrak{B}_{n+1}$,  $\longleftrightarrow$  .

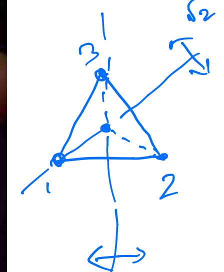
Coxeter groups

In 1935 H. S. M. Coxeter classified all finite groups generated by reflections about hyperplanes (passing through 0) in $\mathbb{R}^n$. Such hyperplanes are called *mirrors* of the group. They include all groups of symmetries of regular convex n-dimensional polytopes and several others. These groups have presentations

$$\langle r_1, r_2, \ldots, r_n | r_i^2 = 1, \quad \underbrace{r_i r_j r_i r_j \ldots}_{m_{ij} \text{ factors}} = \underbrace{r_j r_i r_j r_i \ldots}_{m_{ij} \text{ factors}} \rangle \,,$$

for some integers m_{ij} which form a matrix called the *Coxeter matrix*. Usually the Coxeter matrix is encoded by the *Coxeter–Dynkin diagram* which is a graph whose vertices corresponds to generators, the vertices i and j are adjacent if $m_{ij} \geqslant 3$, an edge is labeled with the value of m_{ij} whenever the value is 4 or greater, and the generators corresponding to non-adjacent vertices commute, $m_{ij} = 2$. Here is the classification of the connected Coxeter–Dynkin diagrams with finite Coxeter group (the subscript of the type indicates the number of vertices which is equal to dimension).

A_2, S_3



$r_1 = (12)$

$r_2 = (23)$

$r_1 r_2 = (123)$

120°

Type	Coxeter–Dynkin diagram	Polytope	Order
$A_n \; (n \geqslant 1)$	•—•—•⋯•—•	n − dimensional symplex	$(n+1)!$
$B_n = C_n \; (n \geqslant 2)$	•—4—•—•⋯•—•	n − dimensional cube	$2^n n!$
$D_n (n \geqslant 4)$	fork •<•—•⋯•—•		$2^{n-1} n!$
E_6	•—•—•—•—• with • below the middle		51840
E_7	•—•—•—•—•—• with • below the third		2903040
E_8	•—•—•—•—•—•—• with • below the third		696729600
F_4	•—•—4—•—•	4D 24-cell	1152
G_2	•—6—•	regular hexagon	12
H_2	•—5—•	regular pentagon	10
H_3	•—5—•—•	icosahedron or dodecahedron	120
H_4	•—5—•—•—•	4D 120-cell or 600-cell	14400
$I_2(m) \quad (m \geqslant 7)$	•—m—•	regular m-gon	$2m$

S_{n+1}

$A_2 = \triangle$

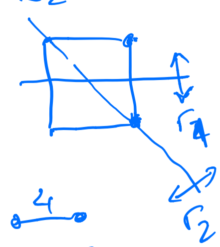
$B_2 = \square$

$H_2 =$

$G_2 =$

$I_2(m)$

B_2

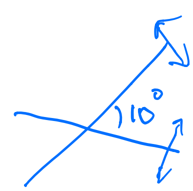


$r_1 r_2 r_1 r_2 = r_2 r_1 r_2 r_1$

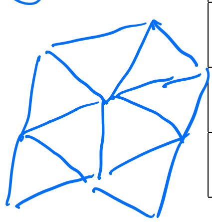
$r_1 r_2 \; 90^\circ$

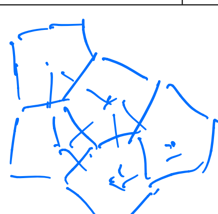
$r_2 r_1 \; 90^\circ$

?



110°





A description of 4D regular polytopes see at https://en.wikipedia.org/wiki/Regular_4-polytope
About the Coxeter groups in general see https://en.wikipedia.org/wiki/Coxeter_group
We may also call $A_2 = I_2(3)$, $B_2 = I_2(4)$, $H_2 = I_2(5)$, $G_2 = I_2(6)$. The Coxeter group of type A_n is isomorphic to the symmetric group S_{n+1}.

Remarkably the same classification appears in many unrelated areas of mathematics. For example, in simple Lie algebras, simple critical points of holomorphic functions, quiver representations, etc.

Artin groups

Braid groups $\mathfrak{B}_n$ were introduced by Emil Artin (the father of Michael Artin whose textbook you probably know from an abstract algebra course) in 1925. An *Artin group* associated with a Coxeter group is obtained from the corresponding Coxeter group by dropping the relation $r_i^2 = 1$. So it has a presentation

$$\langle s_1, s_2, \dots, s_n | \underbrace{s_i s_j s_i s_j \dots}_{m_{ij} \text{ factors}} = \underbrace{s_j s_i s_j s_i \dots}_{m_{ij} \text{ factors}} \rangle \ ,$$

for the entries m_{ij} of the Coxeter matrix. The Artin groups corresponding to the finite Coxeter groups are called *Artin groups of finite type*. We will denote them by the type symbol of the Coxeter group. For instance,

$$A_n := \langle s_1, s_2, \dots, s_n | s_i s_j = s_j s_i \text{ if } |j-i| > 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \rangle = \mathfrak{B}_{n+1} \ .$$

An Artin group is the fundamental group of the space of regular orbits of the complexified action of the corresponding Coxeter group. Namely, we consider a complexification $\mathbb{C}^n$ of $\mathbb{R}^n$ consisting of n-vectors with complex coordinates instead of reals, and allow the elements of the Coxeted group act on $\mathbb{C}^n$ by the same formulas as on $\mathbb{R}^n$. It is known that the space of all orbits of such action is also $\mathbb{C}^n$. The singular orbits Σ are the orbits of points from the mirrors. The complement to Σ in the space of orbits $\mathbb{C}^n$ is called *the space of regular orbits*. The Artin group is the fundamental group of it.

There is an obvious epimorphism of an Artin group to the corresponding Coxeter group generalizing the epimirphism $\mathfrak{B}_n \twoheadrightarrow S_n$

References

[Bir1] J. S. Birman, *Braids, Links and Mapping Class Groups*, Princeton University Press, 1974.

[Mark] A. A. Markov, *Über die freie Aquivalenz geschlossener Zöpfe*, Recueil Mathematique Moscou **1** (1935) 73–78.