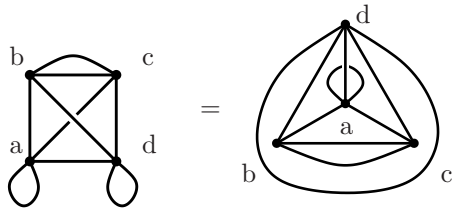


Graphs

Definition. A graph G is a finite set of vertices $V(G)$ and a finite set $E(G)$ of unordered pairs (x, y) of vertices $x, y \in V(G)$ called edges.

A graph may have loops (x, x) and multiple edges when a pair (x, y) appears in $E(G)$ several times. Pictorially we represent the vertices by points and edges by lines connecting the corresponding points. Topologically a graph is a 1-dimensional cell complex with $V(G)$ as the set of 0-cells and $E(G)$ as the set of 1-cells. Here are two pictures representing the same graph.



$$V(G) = \{a, b, c, d\}$$

$$E(G) = \{(a, a), (a, b), (a, c), (a, d), (b, c), (b, c), (b, d), (c, d), (d, d)\}$$

Chromatic polynomial $\chi_G(q)$.

A coloring of G with q colors is a map $\varkappa : V(G) \rightarrow \{1, \dots, q\}$. A coloring \varkappa is proper if for any edge $e: \varkappa(v_1) \neq \varkappa(v_2)$, where v_1 and v_2 are the endpoints of e .

Definition 1. $\chi_G(q) := \#$ of proper colorings of G in q colors.

Properties (Definition 2).

$$\chi_G = \chi_{G-e} - \chi_{G/e};$$

$$\chi_{G_1 \sqcup G_2} = \chi_{G_1} \cdot \chi_{G_2}, \quad \text{for a disjoint union } G_1 \sqcup G_2;$$

$$\chi_{\bullet} = q.$$

Stanley's theorem. For a graph G with n vertices,
 $(-1)^n \chi_G(-1) = \#$ of acyclic orientations of G .

Tutte polynomial $T_G(x, y)$.

Definition 1.

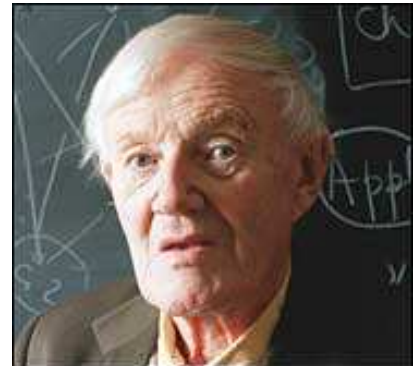
$$T_G = T_{G-e} + T_{G/e} \quad \text{if } e \text{ is neither a bridge nor a loop};$$

$$T_G = xT_{G/e} \quad \text{if } e \text{ is a bridge};$$

$$T_G = yT_{G-e} \quad \text{if } e \text{ is a loop};$$

$$T_{G_1 \sqcup G_2} = T_{G_1 \cdot G_2} = T_{G_1} \cdot T_{G_2} \quad \text{for a disjoint union } G_1 \sqcup G_2 \text{ and a one-point join } G_1 \cdot G_2;$$

$$T_{\bullet} = 1.$$



Properties.

$$T_G(1, 1) \quad \text{is the number of spanning trees of } G;$$

$$T_G(2, 1) \quad \text{is the number of spanning forests of } G;$$

$$T_G(1, 2) \quad \text{is the number of spanning connected subgraphs of } G;$$

$$T_G(2, 2) = 2^{|E(G)|} \quad \text{is the number of spanning subgraphs of } G;$$

$$\chi_G(q) = q^{k(G)} (-1)^{r(G)} T_G(1 - q, 0);$$

Definition 2.

Let \bullet F be a graph;

- $v(F)$ be the number of its vertices;
- $e(F)$ be the number of its edges;
- $k(F)$ be the number of connected components of F ;
- $r(F) := v(F) - k(F)$ be the rank of F ;
- $n(F) := e(F) - r(F)$ be the nullity of F ;

$$T_G(x, y) := \sum_{F \subseteq E(G)} (x - 1)^{r(G) - r(F)} (y - 1)^{n(F)}$$

Dichromatic polynomial $Z_G(q, v)$ (**Definition 3**).

Let $Col(G)$ denote the set of colorings of G with q colors.

$$Z_G(q, v) := \sum_{\kappa \in Col(G)} (1 + v)^{\# \text{ edges colored not properly by } \kappa}$$

Properties .

$$Z_G = Z_{G-e} + vZ_{G/e} ;$$

$$Z_{G_1 \sqcup G_2} = Z_{G_1} \cdot Z_{G_2} , \quad \text{for a disjoint union } G_1 \sqcup G_2 ;$$

$$Z_{\bullet} = q ;$$

$$Z_G(q, v) = \sum_{F \subseteq E(G)} q^{k(F)} v^{e(F)} ;$$

$$\chi_G(q) == Z_G(q, -1) ;$$

$$Z_G(q, v) = q^{k(G)} v^{r(G)} T_G(1 + qv^{-1}, 1 + v) ;$$

$$T_G(x, y) = (x - 1)^{-k(G)} (y - 1)^{-v(G)} Z_G((x - 1)(y - 1), y - 1) .$$

Potts model in statistical mechanics (Definition 4).

Potts model (C.Domb 1952); $q = 2$ the Ising model (W.Lenz, 1920)

Let G be a graph.

Particles are located at vertices of G . Each particle has a *spin*, which takes q different values . A *state*, $\sigma \in \mathcal{S}$, is an assignment of spins to all vertices of G . Neighboring particles interact with each other only if their spins are the same.

The energy of the interaction along an edge e is $-J_e$ (*coupling constant*). The model is called *ferromagnetic* if $J_e > 0$ and *antiferromagnetic* if $J_e < 0$.

Energy of a state σ (*Hamiltonian*),

$$H(\sigma) = - \sum_{(a,b)=e \in E(G)} J_e \delta(\sigma(a), \sigma(b)).$$

Boltzmann weight of σ :

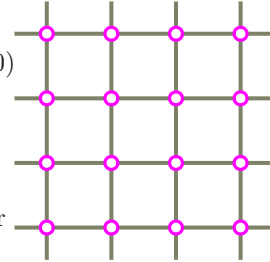
$$e^{-\beta H(\sigma)} = \prod_{(a,b)=e \in E(G)} e^{J_e \beta \delta(\sigma(a), \sigma(b))} = \prod_{(a,b)=e \in E(G)} \left(1 + (e^{J_e \beta} - 1) \delta(\sigma(a), \sigma(b)) \right),$$

where the *inverse temperature* $\beta = \frac{1}{\kappa T}$, T is the temperature, $\kappa = 1.38 \times 10^{-23}$ joules/Kelvin is the *Boltzmann constant*.

The Potts partition function (for $x_e := e^{J_e \beta} - 1$)

$$Z_G(q, x_e) := \sum_{\sigma \in \mathcal{S}} e^{-\beta H(\sigma)} = \sum_{\sigma \in \mathcal{S}} \prod_{e \in E(G)} (1 + x_e \delta(\sigma(a), \sigma(b)))$$

Properties of the Potts model Probability of a state σ : $P(\sigma) := e^{-\beta H(\sigma)} / Z_G$.



Definition.

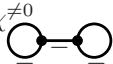
$\chi_G(2q + 1) := \#$ of proper q -colorings of G .


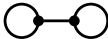
$\chi_G^{\neq 0}(2q) := \#$ of proper q -colorings of G which take nonzero values.

Properties.

- $\chi_G(\lambda)$ is a polynomial function of $\lambda = 2q + 1 > 0$;
- $\chi_G^{\neq 0}(\lambda)$ is a polynomial function of $\lambda = 2q > 0$;
- $\chi_G(\lambda) = \chi_{G-e}(\lambda) - \chi_{G/e}(\lambda)$;
- $\chi_G^{\neq 0}(\lambda) = \chi_{G-e}^{\neq 0}(\lambda) - \chi_{G/e}^{\neq 0}(\lambda)$;
- $\chi_{G_1 \sqcup G_2} = \chi_{G_1} \cdot \chi_{G_2}$ and $\chi_{G_1 \sqcup G_2}^{\neq 0} = \chi_{G_1}^{\neq 0} \cdot \chi_{G_2}^{\neq 0}$ for a disjoint union $G_1 \sqcup G_2$;
- $\chi_{\emptyset} = 1$.

Example. $\chi^{\neq 0}(2q) = 2q(2q - 1)$.



There are two tricky issues in Zaslavky’s acyclicity theorem. The first one is the notion of a cycle. A subset S of edges of a sign graphs is called *balanced* if for every circuit in S the product of the signs of edges of the circuit is equal to 1. A *cycle* of a signed graph G is a subgraph of one the following 3 types: 1) a balanced circuit, 2) a subdivision of a tight handcuff  with both circuits to be unbalanced, and 3) a subdivision of a loose handcuff  with both circuits to be unbalanced. The second issue is a notion of orientation. An *orientation* of an edge is a pair of arrows on its half-edges which are coherent for positive edges and not coherent for negative edges. An orientation of a sign graph is *acyclic* if every cycle contains either a source or a sink. An orientation of a sign graph is *compatible* with a coloring c if for every positive edge the color of it arrow-head is greater or equal to the color of it arrow-tail and for every negative edge the sum of colors of its ends is not negative (resp. not positive) for the arrows pointed towards the ends (resp. away from the end).

Theorem. [Za, Theorem 3.5] *Let $q \in \mathbb{N}$ and G be a signed graph with n vertices. The number of compatible pairs of acyclic orientations of G and colorings $V(G) \rightarrow \{-q, -q + 1, \dots, -1, 1, \dots, q - 1, q\}$ is equal to $(-1)^n \chi_G^{\neq 0}(-2q)$.*

Example. For $q = 1$ and the graph above we have $\chi^{\neq 0}(-2) = 6$. Here are 6 compatible

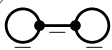
pairs of acyclic orientations and colorings $V(G) \rightarrow \{-1, 1\}$ (we mark the source-vertex red and the sink-vertex blue).



Note that the first two colorings are proper and the last four are improper.

B-symmetric chromatic function of signed graphs. [Ra, Ch]

$$Y_G(\dots, x_{-2}, x_{-1}, x_1, x_2, \dots) := \sum_{\substack{\varkappa: V(G) \rightarrow \mathbb{Z} \setminus \{0\} \\ \text{proper}}} \prod_{v \in V(G)} x_{\varkappa(v)}$$

Example. Y  = \dots
 $x_{-2}(\dots x_{-2} + x_{-1} + x_1 + \widehat{x_2} + \dots)$
 $x_{-1}(\dots x_{-2} + x_{-1} + \widehat{x_1} + x_2 + \dots)$
 \dots
 $= p_{1,0}^2 - p_{1,1}$, where

$p_{a,b} := \sum_{i \in \mathbb{Z} \setminus \{0\}} x_i^a x_{-i}^b$ are the *signed power functions*.

The signed chromatic polynomial $\chi_G^{\neq 0}(2q)$ is a specialization of Y_G obtained by substitution $x_i = 1$ for $|i| \leq q$ and $x_i = 0$ for $|i| \geq q$. This is the same substitution as $p_{a,b} = \lambda = 2q$.

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