Knots and Graphs

Working Group [Summer 2020]

MATH 4193, class number 16355

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RESEARCH PROJECTS

Project 1. ***Symmetric Tutte polynomial.*** (Oscar Coppola, Torey Hilbert, Aditya Jambhale, Alexander Patterson)

R.Stanley introduced the *Stanley's chromatic symmetric function* in [St1] as a generalization of classical chromatic polynomial of a graph. It is a formal power series in countably many variables $x_1, x_2, \ldots$ associated with a graph. It is symmetric under the permutation of variables. As such it can be expressed in terms of the power functions $p_n = \sum x_i^n$. For a given graph only finitely many variables p_n participate in the this expression, so it becomes a polynomial in variables p_n. A specialization to all variables to a single variable $p_n = q$ gives the classical chromatic polynomial.

The Tutte polynomial is another generalization of the classical chromatic polynomial to a polynomial in two variables x and y. A remarkable property of the Tutte polynomial is that for dual planar graphs the Tutte polynomials differ by interchanging x and y.

The goal of the project is to try to find a symmetric generalization of the Tutte polynomial for planar graphs which would be a polynomial in two collection of variables p_n and q_n, such that for dual planar graph these variables would be swapped.

Project 2. ***Acyclic orientations of signed graphs.*** (Eric Fawcett, Kat Husar, Hannah Johnson, Michael Reilly)

This project is related to Project 1. A classical Stanley's theorem claims that the evaluation of the chromatic polynomial at -1 is equal to the number of acyclic orientations of the graph. It has a generalization [St1, Theorem 3.3] to the symmetric chromatic function from Project 1.

From the other hand, there is a generalization [Za] of the chromatic polynomial to *signed graphs*, where a sign ± 1 is assigned to every edge of a graph. Zaslavsky also generalized [Za, Theorem 3.5] Stanley's acyclicity theorem to signed graphs.

Stalney's chromatic symmetric function was generalized to signed graphs in the previous years programs (see slides of presentations [Ra, Ch]). The goal of the project is to find a symmetric generalization of [Za, Theorem 3.5] which would be an analog of [St1, Theorem 3.3].

Project 3. ***Braid groups.*** (Vlad Akavets, Jay Patel, Caitlin Patterson, John White)

The braid group on n strands [KT] is given by generators and relations

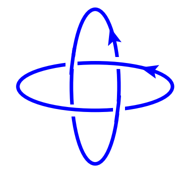
$$\mathfrak{B}_n := \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |j-i| > 1, \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle$$

It is well known that $\mathfrak{B}_3$ is the only braid group which is isomorphic to the fundamental group of a knot complement. However the sequence of braid groups can be generalized to the Artin groups given by generators and relations

$$\langle s_1, s_2, \ldots, s_n \mid \underbrace{s_i s_j s_i s_j \ldots}_{m_{ij} \text{ factors}} = \underbrace{s_j s_i s_j s_i \ldots}_{m_{ij} \text{ factors}} \rangle ,$$

for some integers m_{ij}.

The goal of the project is searching for links whose group would be an Artin group. For example, for the Solomon link on the right (http://katlas.math.toronto.edu/wiki/L4a1), the fundamental group of its complement is the Artin group



$$B_2 = \langle s_1, s_2 \mid s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 \rangle$$

with $m_{12} = 4$. Perhaps we need to take into account the symmetry of a link and then consider an *orbifold fundamental group*. Another development in this direction could be to consider *virtual* knots/links and virtual braids.

Project 4. ***Thompson groups.*** (Jake Huryn, Rushil Raghavan, Dennis Sweeney)
A Thompson group F is a group of piecewise linear homeomorphism of $[0, 1]$ with breaks at dyadic rational numbers and such that on each interval of linearity its derivatives are powers of 2. (see [CF, CFP]). Every element of F can be encoded by a pair of binary trees with the same number of vertices. V. Jones [Jo] find a way to construct a link for every such element and proved that every link can be constructed in this way (see also [Ai]). In this project we will try find a Markov's types theorem for such a presentation. That is we will try to describe any two elements of F with produce the same link.

References

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[Ch] S. Chmutov, *B-symmetric chromatic function of signed graphs*, https://people.math.osu.edu/chmutov.1/talks/2020/slides-Moscow.pdf Video is available at https://www.youtube.com/watch?v=khA7rP84sYY

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[St1] R. Stanley, *A symmetric function generalization of the chromatic polynomial of a graph*, Advances in Math. **111**(1) (1995) 166–194.

[Za] T. Zaslavsky, *Signed graph coloring*, Discrete Mathematics **39**(2) (1982) 215–228.