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Khovanov Homology of Knots

Smoothings

Given $X$, $\gamma$ is the 0 smoothing, and $\gamma'$ is the 1 smoothing.

If $X$ is the set of all crossings of $L$, a link, then $\Delta \in \gamma$ corresponds to a "complete smoothing" $S^\gamma$ of $L$, a state. The Jones Polynomial is

$$\frac{1}{2} \sum_{m\in \gamma} (-1)^m q^m \left\{ (q + q^{-1})^m \right\}$$

where $\gamma$ is the height, i.e. $\#$ of $1$-smoothings, and $k(\gamma)$ is the number of cycles in $S^\gamma$, where $n = |X|$ and $n' = \#$ of right hand crossings, $n'' = \#$ of left hand crossings.

Example: The Trefoil

$$q(q+q^{-1})$$

$$q^2(q+q^{-1})^2$$

$$q^3(q+q^{-1})^3$$

$$(q+q^{-1})^2 - 3q(q+q^{-1}) + 3q^2(q+q^{-1})^2 - 3q^2(q+q^{-1})^3$$

$$= q^{-2} + 1 + q^2 - q^6 \xrightarrow{\frac{1}{n+1} - \frac{1}{n''} - \frac{1}{n'} \geq (2,0)} q + q^2 + q^5 - q^7(q+q^{-1})^{-1}$$

$$J(S^2) = q^2 + q^6 - q^8.$$
Aside: Euler characteristic and (singular) homology.

Euler characteristic defined as: \( \chi(K) = \sum_i (-1)^i \text{rank}(C_i(K)) \)

Theorem: \( \chi(K) = \sum_i (-1)^i \text{rank} H_i(K) \) (Munkres, Thm 22.2)

The relationship between Euler characteristic and (singular) homology is analogous to Jones polynomial and Khovanov Homology.

Suppose \( W = \bigoplus W_m \) is a graded vector space.

Definition (graded dimension) \( \text{qdim} W = \sum_m q^m \dim W_m \)

Definition (degree shift) \( \mathcal{E} \) is the degree shift operation. Suppose \( W = \bigoplus W_m \). Then \( W \mathcal{E} = \bigoplus W_{m+1} \).

Example \( \mathcal{E} \) qdim \( W \mathcal{E} = q^1 \cdot \text{qdim} W \).

Definition (height shift) Suppose \( \overline{C} \) is a chain complex: \( \ldots \rightarrow \overline{C}^n \xrightarrow{d^n} \overline{C}^{n+1} \xrightarrow{d^{n+1}} \ldots \)

where \( \overline{C}^n \) are (graded) vector spaces. \( h \) is the height of \( \overline{C}^n \). Let \( \overline{C} = \overline{C}^{\geq h} \).

Then \( \overline{C}^n = \overline{C}^{n-h} \) and the differentials shift similarly.
Now, I'll construct the chain complex for the Khovanov homology.

Let $V$ be the graded vector space with two basis elements: $v_+$ and $v_-$ whose degrees are $+1$, $-1$ resp.

To each $x \in \mathbb{Z}_0, 1^2 \mathbb{Z}$ associate the graded vector space $V_x(L) := V^\otimes k \mathbb{Z} \mathbb{Z}$ ($k$ = # of cycles, $r = |a| = \pm a_i$)

**Definition**

The chain group $[L]^r$ ($0 \leq r \leq n$) is the direct sum of all vector spaces at height $r$: $[L]^r := \bigoplus_{a : r = K1} V_x(L)$. The graded object is

$$C(L) := \bigoplus [L]^{[-n-] \mathbb{Z} n_+ - 2n_-, [L]}$$

For the trefoil we have

$$\begin{array}{cccc}
V^\otimes 2 & \oplus & V^\otimes 1^2 \mathbb{Z} & \oplus \ V^\otimes 2 \mathbb{Z} \mathbb{Z} \\
[100] & \rightarrow & [8] & \rightarrow [8] \end{array}$$

$$\begin{array}{c}
\frac{[-n-] \mathbb{Z} n_+ - 2n_-}{(n_+, n_-) = (3, 0)} \rightarrow C(\circ)
\end{array}$$

Check: $\chi(L) \mathbb{Z}, (-1)^q \dim [L]^r = \hat{1}(L)$
We can label the edges of \( \mathcal{E}_0, 1^3 \times \) by sequences in \( \mathbb{Z}_0, 1, \star \times )^3 \) where only one \( \star \) appears.

The tail of the edge is the sequence with \( \star \to 0 \) at vertex \( u \)

and the head is the sequence \( \star \to 1 \).

i.e. \((00\star) \rightarrow 001 \).

Call the edge (or rather map) \( \delta \) and

\[ d^\pm = \sum (-1)^o d^o \delta. \]

The sign of \((-1)^o \) will be determined by first choosing them all to be positive. This makes the map commutative. Then making a subset \((-1)^o \) will make it anti-commutative — hence \( d^o d^o = 0 \).

**Defining \( d^o \):**

Now, \( \delta \) must be defined to make the map commute. By observation, only changing one crossing will do one of two things to the smoothing:

(i) \( (00 \to \infty ) \) or \( V \otimes V \to V \)

(ii) \( (\infty \to 0 0 ) \) or \( V \to V \otimes V \)

\[ m : \begin{cases} V_+ \otimes V_- & \to V_- \\ V_- \otimes V_+ & \to V_+ \end{cases} \]

\[ \Delta : \begin{cases} V_+ & \to V_+ \otimes V_- + V_- \otimes V_+ \\ V_- & \to V_- \otimes V_- \end{cases} \]